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Overreactions, Momentum, Liquidity, and Price Bubbles in Laboratory and Field Asset Markets

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Laboratory asset markets provide an experimental setting in which to observe investor behavior. Over more than a decade, numerous studies have found that participants in laboratory experiments frequently drive asset prices far above fundamental value, after which the prices crash. This bubble-and-crash behavior is robust to variations in a number of variables, including liquidity (the amount of cash available relative to the value of the assets being traded), short-selling, certainty or uncertainty of dividend payments, brokerage fees, capital gains taxes, buying on margin, and others.

This paper attempts to model the behavior of asset prices in experimental settings by proposing a "momentum model" of asset price changes. The model assumes that investors follow a combination of two factors when setting prices: fundamental value, and the recent price trend. The predictions of the model, while still far from perfect, are superior to those of a rational expectations model, in which traders consider only fundamental value. In particular, the momentum model predicts that higher levels of liquidity lead to larger price bubbles, a result that is confirmed in the experiments. The similarity between laboratory results and data from field (real-world) markets suggests that the momentum model may be applicable there as well.

What drives stock prices? Certainly earnings—or more generally, fundamental value—play a role. Investor behavior, however, is increasingly considered as another important factor. Unfortunately, it is difficult to observe most of the important variables in actual trading behavior. Experimental asset markets seek to overcome this limitation by observing traders in a controlled setting, where fundamental value is a known quantity (an impossibility in field markets), and where other sources of uncertainty can be eliminated as well.

In a typical laboratory experiment, a group of subjects participates in a trading session lasting approximately two hours. Each subject is given a certain amount of cash and stock with which to trade. Trading is conducted by computer over a local network, and the session is divided into fifteen periods. In each period, bids and offers are matched and the trades clear simultaneously. The subjects are told at the beginning that their shares will pay a dividend each period. The amount of each dividend is not known in advance, but the subjects are told what distribution the dividend payments are drawn from. For example, there may be a one in four probability of either 60 cents, 28 cents, 8 cents, or zero. The average (expected) dividend would then come out to 24 cents per period. This means that the fundamental value of each share is $3.60 for the full fifteen periods. After each period, the value drops by 24 cents. The participants are told this at the beginning, so they know the fundamental value of their shares.

In laboratory experiments, no assumptions are made regarding traders' decision-making processes. This can be compared to rational expectations theory, for example, where there are very definite assumptions that investors are unbiased processors of information. In either case, there are two important variables: the information that bears on fundamental value, and investor behavior. In field markets, information is incomplete, and it is not always clear whether fundamental value or investor behavior is the most important determinant of stock prices, because both tend to be highly uncertain. Traditionally, finance theory and research have, in effect, treated investor behavior as known and fixed, while taking fundamental value to be the only source of uncertainty. By observing traders in a laboratory market setting, we can turn this situation around and control for variations and uncertainty in fundamental value, thus allowing for a relatively clear observation of investor behavior.

We begin by examining the database of laboratory experiments in which full information on the dividend distribution (including the calculated value of expected dividend value each period) is provided to all subjects (second through fourth sections). The effects of a large

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number of treatments, subject pools, subject experience, and institutional variations (brokerage fees, capital gains taxes, short-selling, margin buying, futures contracting, limit price change rules—circuit breakers—and call market organization) are provided. Subject experience, subject sophistication, and futures contracting are the only treatments that materially dampen the robust tendency of the stock markets in these environments to produce price bubbles and crashes relative to fundamental value.

In the fifth and sixth sections, we articulate mathematically a "momentum" model that modifies the rational expectations approach by postulating that investor sentiment is of two types: 1) fundamentalists whose purchases are positively (negatively) related to the discount (premium) of price relative to intrinsic values; and 2) momentum traders whose purchases are positively related to the percentage rate of change in price. The laboratory research findings and the model are then used to interpret field data sets, including two examples of bubbles in closed-end funds (where fundamental value is known, and this information is widely available), two funds with identical portfolios, and the frequency of crashes in the Standard & Poor's index.

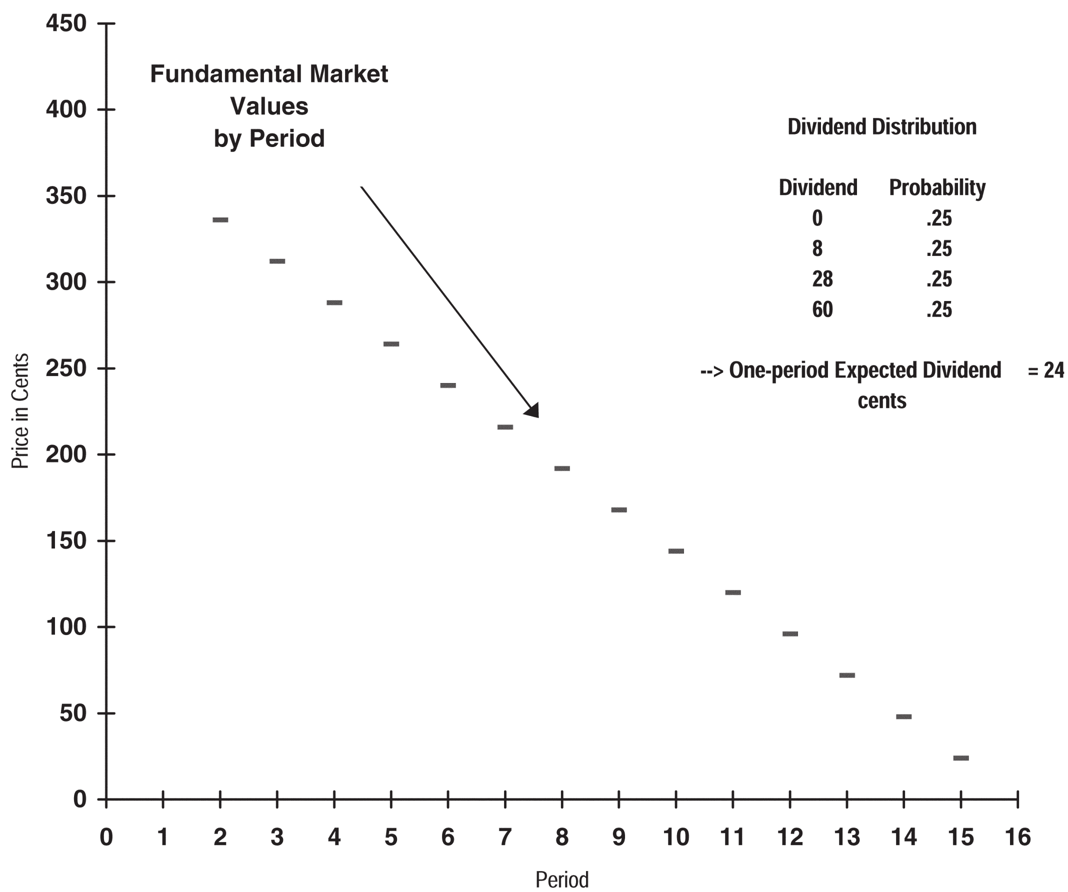
The seventh section then compares various methods of predicting laboratory stock market prices. In particular, we compare the momentum model, expert trader forecasts, time series forecasts, and a price adjustment process in which price changes are a linear function of the excess bids (bids versus asks—similar to a Walrasian price adjustment process where price adjusts in the direction of excess demand).

Finally, in the last section, we interpret the momentum model parametrically in terms of a measure of market liquidity, which can be controlled in the laboratory as a treatment variable, and report a series of experiments based on the liquidity interpretation of the momentum model. Contrary to the rational expectations model, this model predicts that asset prices will be an increasing function of the aggregate ratio of cash to share endowments, a result that is corroborated by the liquidity experiments.

Basic Bubble Experiments

An experimental design for studying the temporal evolution of asset trading prices was introduced by Smith, Suchanek, and Williams (SSW) [1988]). Figure 1

FIGURE 1
Baseline Asset Market Experiment Parameters



illustrates one of their experimental environments. Each of the twelve traders receives an initial portfolio of cash and shares of a security with a fixed life of fifteen trading periods. Before each trading period, t = 1, 2, …, 15, the expected dividend value of a share,[1] $0.24(15 – t + 1), is computed and reported to all subjects to guard against any misunderstanding. This situation is like that of a stock mutual fund, whose net asset value is reported to investors daily or weekly. Each trader is free to trade shares of the security using double auction trading rules (see Williams [1980] for details of these rules), which are similar to those used on major stock exchanges. At the end of the experiment, a sum equal to all dividends received on shares, plus initial cash, plus capital gains, minus capital losses, is paid in U.S. currency to each trader.

The rational expectations model predicts that prices track the fundamental value line (see Figure 1). Behaviorally, however, inexperienced traders produce high *amplitude*[2] bubbles that can rise two to three times above fundamental value. In addition, the span of a boom tends to be of long *duration* (ten to eleven periods), with a large *turnover* of shares (five to six times the outstanding stock of shares over the fifteen-period experiment). In nearly all cases, prices crashed to fundamental value by period 15.

Figure 2 contrasts the mean contract prices and volume for inexperienced traders with those for experienced traders in two laboratory asset markets. The data points plot the mean price for each period and the numbers next to the prices show the number of contracts made in that period. With inexperienced traders, bubbles and crashes are standard fare, but this phenomenon disappears as traders become experienced. That is, traders twice experienced in trading in a laboratory asset market will trade at prices that deviate little from fundamental value.

The robustness of the bubble/crash phenomenon has led several researchers to examine changes in the basic trading environment and rules to see if such changes can reduce or eliminate this large systematic price deviation from fundamental value. We describe below the research testing various hypotheses that might contribute to, or retard, the formation of price bubbles (for a survey of the literature on asset trading experiments, see Sunder [1995]).

Changes in the Economic Environment

Recall that in the baseline experiments individual traders were endowed with different initial portfolios (see footnote 1). A common characteristic of first-period trading is that buyers tend to have low share endowments, while sellers have relatively high share endowments. Based on conventional utility theory, risk-averse traders might be using the market to acquire more balanced portfolios. If diversification preferences account for the low initial prices, which in turn leads to arbitrage that creates expectations of further price increases, making the initial trader endowments equal across subjects would dampen bubbles. However, *observations from four experiments with inexperienced traders show no significant effect of equal endowments on bubble characteristics* (see King, Smith, Williams, and Van Boening (KSWV) [1992]). Thus, the conjecture that initial portfolio rebalancing depresses prices, with subsequent price increases leading to expectations of capital gains, cannot be substantiated.

FIGURE 2
Mean Contract Price and Total Volume

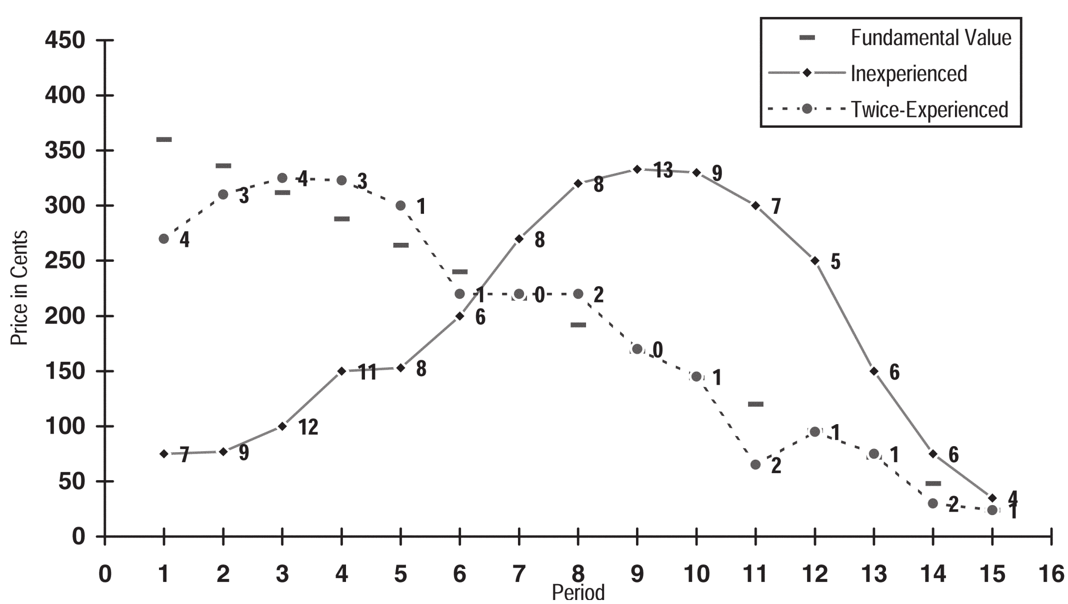
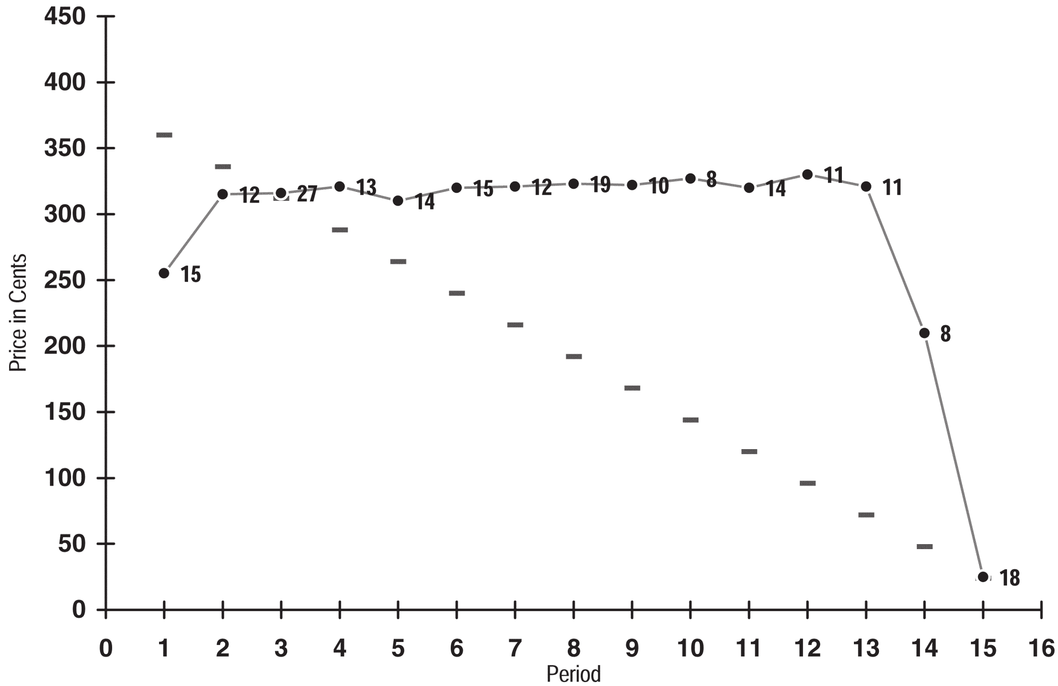


FIGURE 3
Mean Contract Price and Total Volume: Certain Dividend



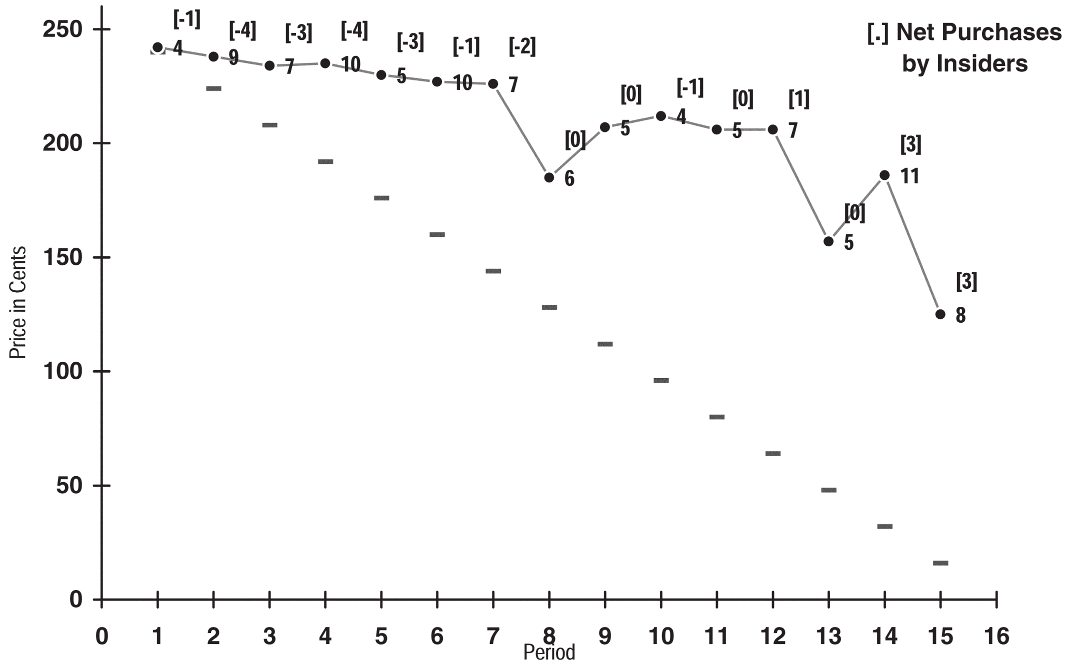
A second conjecture based on risk aversion deals with price expectations due to dividend uncertainty. In this case, it may be that a divergence of expectations concerning dividends can cause price increases in early periods when the cumulative dividend variance is largest (the single-period variance is 26.73 cents, so the variance of the sum of dividends at period t is $[15-(t-1)] \times 26.73$ cents). The elimination of such uncertainty should reduce the severity of bubbles if this conjecture is true.

Experiments by Porter and Smith (PS) [1995] show that the elimination of dividend uncertainty is not a sufficient condition to eliminate bubbles (see Figure 3 for an example). In particular, *when the dividend draw each period is set equal to the one-period expected dividend value, so that the asset dividend stream is certain, bubbles still occur and are not significantly different from the case with dividend uncertainty*. This is consistent with the hypothesis that an important factor in the occurrence of bubbles is traders' uncertainty about the behavior of other traders. The bubble is all but eliminated, however, when dividends are certain and subjects are more experienced, which suggests that dividend certainty assists traders in achieving common expectations of value.

Lei, Noussair, and Plott [1998] investigate the capital gains expectation motivation for bubbles through a series of experiments that try to eliminate this motivation. In one treatment, they restrict the trading mechanism by not allowing reselling, so that the ability to capture capital gains is eliminated. They restrict the role of each subject to that of *either* a buyer or a seller; this artificial restriction eliminates the ability of any subject to buy for the purpose of resale. *They find that they are able to reproduce the empirical patterns of previous bubble experiments.*

Rational expectations theory predicts that anyone aware of the tendency of traders to overreact in these markets could engage in profitable arbitrage. Thus, knowledgeable traders will take advantage of these opportunities, thus dampening the price volatility in these markets. KSWV test this hypothesis by creating a set of "insider" traders. Specifically, three graduate students read the SSW paper, and were given data on the performance of a group of inexperienced undergraduates, who returned for a second session as uninformed but experienced subjects. The graduate students then participated in this session as informed "insiders" and were given summary information on the number of bids and offers entered at the end of each period (SSW showed that the excess number of bids over offers in a period was positively correlated with the change in mean price from the current to the next period; see section VIIB). These informed subjects participated in markets with six or nine uninformed traders recruited as above. In addition to having the same share endowments as the uninformed traders, the informed traders each had a capacity to sell shares borrowed from the experimenters. These short sales had to be repurchased and returned to the experimenter before the close of period 15. While the results provide support for *the rational expectations prediction when the uninformed subjects are experienced, as described above, when the uninformed traders are inexpe-*

FIGURE 4
Mean Contract Price, Volume, and Insider Purchases



rienced, the bubble forces are so strong that the insiders are swamped by the buying wave. By period 11, the insiders reached their maximum selling capacity, including short sales[3] *(see Figure 4).*

Note that short sales not covered by purchases were exposed to a $1.20 penalty per share (half the first-period dividend value of $2.40). When facing inexperienced traders, in Figure 4, short covering by expert traders prevented the market from crashing to dividend value in period 15.[4] Thus, short-selling against the bubble, while tending to dampen the bubble, prevented convergence to the rational expectations value at the end.

Finally, a fairly typical criticism of using experimental evidence to test economic theory is that the subjects in experiments are usually inexperienced college undergraduates. While most theories do not distinguish among various demographic or "experience" factors, it is assumed that the theory applies only to "sophisticated" traders. The problem with testing this proposition is that it does not specify in advance the characteristics of the appropriate subjects, so that if a test of the theory yields negative results, one can always conclude that subjects were not sophisticated enough.

However, one can ask: Could the use of professional traders or business executives eliminate this uncertainty concerning the rationality of others' behavior? The experimental answer to this question shows: *The use of subject pools of small businesspeople, mid-level corporate executives, and over-the-counter market dealers has no significant effect on the characteristics of bubbles with first-time subjects.* In fact, one of the most severe bubbles among the original twenty-six SSW experiments occurred when using small businessmen and women from the Tucson, Arizona community. Subsequently, we conducted experiments each year using mid-level executives enrolled in the Arizona Executive Program (for comparison with a "less sophisticated" group of subjects, see Figure 2). *All* of those experiments produced bubbles. Figure 5 charts mean prices and volumes by period for subjects enrolled in one of the Arizona Executive Programs.

It has also been shown that advanced graduate students in economics, trained in economic and game theory but inexperienced in laboratory asset trading, also trade at prices that track fundamental value very closely. Figure 6 plots the result for a group of such graduate students from leading American universities who participated in a workshop in experimental economics. Mean prices for these subjects were always within 5 cents of fundamental value. Also shown in Figure 6 are the contrasting data from a typical group of undergraduates.[5]

Institutional Treatments

In the above experiments, many features common in field asset markets are absent: brokerage fees, capital gains taxes, short sales, margin purchases, futures contracting, and circuit breaker regulations. In this section, we report experiments that introduce these features and determine their impact on trading behavior. We also report the effect on the market organization when the continuous trading system is replaced by a

FIGURE 5
Mean Contract Price and Volume, Arizona Executives

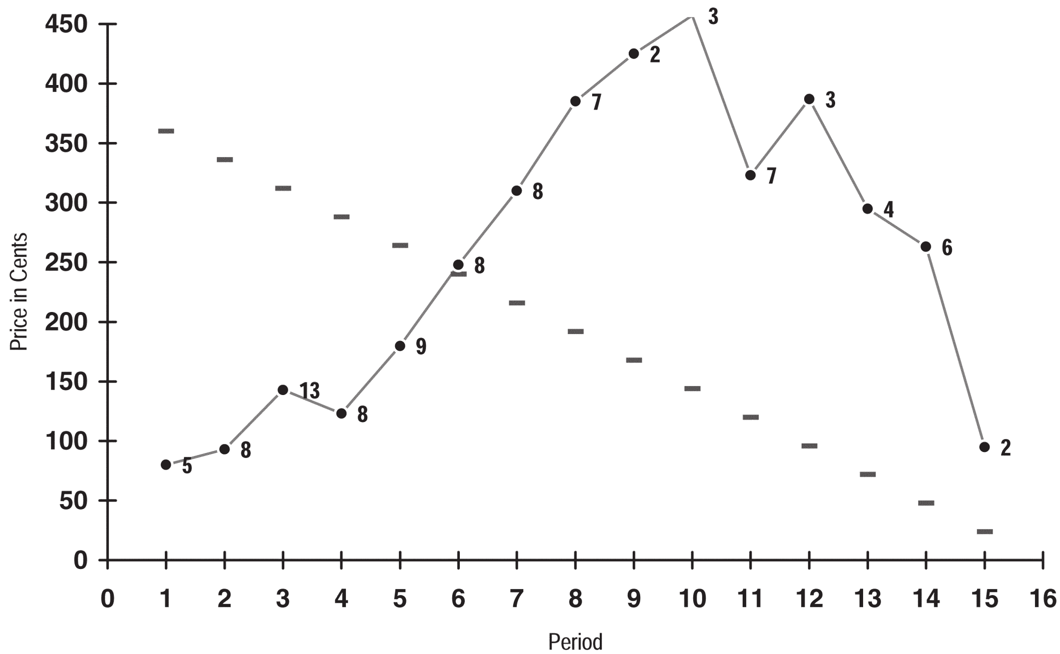
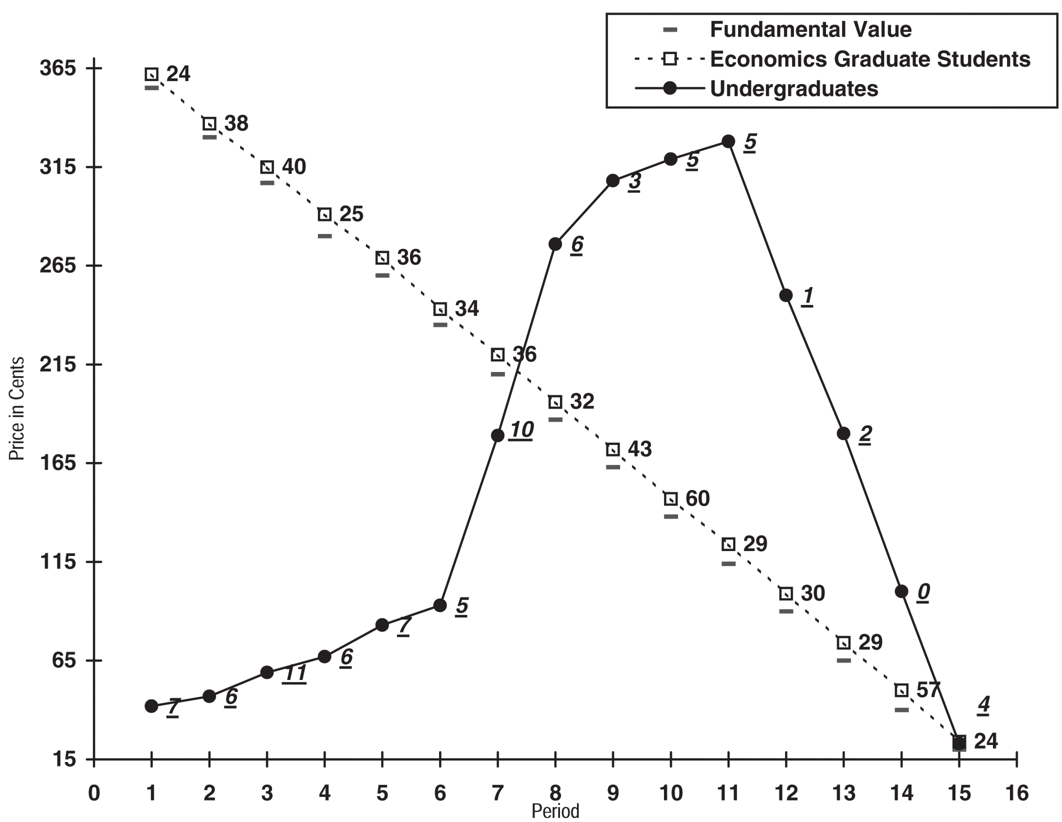


FIGURE 6
Mean Contract Price and Volume for Advanced Economics Graduate Students Versus UA Undergraduates



call market in which all orders are aggregated at the beginning of a period and all trades are made at the call with one market clearing price. We conducted experiments on each of these features individually. We did not conduct any experiments that included all the changes (transaction costs, certain dividend, futures, short-selling, insider, and so on). From the data, we suspect that futures, short-selling, and insiders have a strong effect on reducing the magnitude of the bubble, while the effect of the others is neutral, and margin buying only makes things "worse."

Brokerage Fees/Capital Gains Taxes

The market used to trade assets in these experiments has low participation costs of trading, since subjects only have to touch a button to accept standing bids or asks (the same is essentially true now of trading via online brokers). This, coupled with the conjecture that laboratory subjects may believe they are expected to trade, may result in price patterns that deviate from rational expectations equilibrium. One way to test the transactions cost hypothesis is to impose a fee on each exchange. The addition of this brokerage fee has very little effect on bubble characteristics. In particular, *a 20 cent fee on each trade (10 cents each on the buyer and seller) had no significant effect on the amplitude, duration, or share turnover.*

In addition to transaction fees, bubbles may form due to capital gains expectations and the "greater fool" theory. To dampen this form of price expectations formation, Lei, Noussair, and Plott [LNP] [1998] impose a capital gains tax of 50% on all traders. *They find that the capital gains tax does not reduce the tendency for bubbles to occur. Either other factors account for bubbles, or capital gains expectations are strong enough to overcome reductions in their profitability.*

Contracting Forms (Short Sales, Margin Buying, and Futures)

If individual traders could take a position on either side of the market and leverage their sales by selling borrowed shares (taking a short position), or leverage their purchases by buying with borrowed funds (margin buying), it is conjectured that traders who believe prices should be at fundamental value can offset the overreaction of other traders. KSWV conducted several experiments in which subjects were given a zero interest loan, with principal repaid at the end of the experiment so that margin buying was possible. In addition, subjects were also given the ability to sell borrowed shares that had to be returned by the end of the experiment. Neither condition, margin funds or the ability to sell short, is sufficient to reduce bubble characteristics; in the case of margin buying, the bubble becomes worse. *Margin buying opportunities cause a significant increase in the amplitude of bubbles for inexperienced traders. Short-selling does not significantly diminish the amplitude and duration of bubbles, but the volume of trade is increased significantly. Figure 7 provides an example.*

Figure 7 highlights the problem of the timing of short sales. In periods 6, 7, and 10, net short sales are negative, indicating net purchases to cover short positions. In period 13, net short sales are zero. All these covering purchases are at prices near the peak of the bubble, and therefore tend to exacerbate the bubble. But the traders could not know this, and behaved as if prices would continue upward.

One major criticism of the contracting form used in the original experiments of SSW is that traders *cannot* obtain information about potential future prices through the market. Specifically, traders must form price expectations internally, without any market means to calibrate their expectations. In the field, traders have access to prices in futures markets to help them hedge risks and to get a market reading on future price expectations. A futures market could provide immediate feedback to traders who can see that the bubble is not likely to persist, and thus allow ebullient expectations to unravel.

To test this hypothesis, PS ran two sequences of two experiments with subjects who were first trained in the mechanics of a futures market. In the training sequences, subjects participated in a series of two-period markets, with futures contracts in period 1 maturing in period 2. In this manner, subjects learned that a futures contract is equivalent to a cash contract in the period in which it expires, and should trade at the same price. In the new treatment experiments, a futures contract expiring in period 8 was used, and agents could trade both the spot and the period 8 futures contracts in periods 1–8; after period 8, only the spot market was active. This contracting regime provides observations (futures' contract prices) on the group's period 8 expectations during the first seven periods of the market.

Figure 8 shows the results of one of these futures market experiments (the other experiment did not converge to dividend value in period 8, but produced a smaller bubble than is common without a futures market). In particular, *futures markets dampen, but do not eliminate, bubbles by speeding up the process by which traders form common expectations*. Note that the spot market trades at mean prices less than fundamental value for the first seven periods, while the futures market trades at, or under, the period 8 share value for the first seven periods. But the trades are minimally rational in the sense that spot shares trade at prices above the futures prices (spot shares have higher dividend values than a future on period 8).

FIGURE 7
Mean Contract Price, Volume, and Short Sales

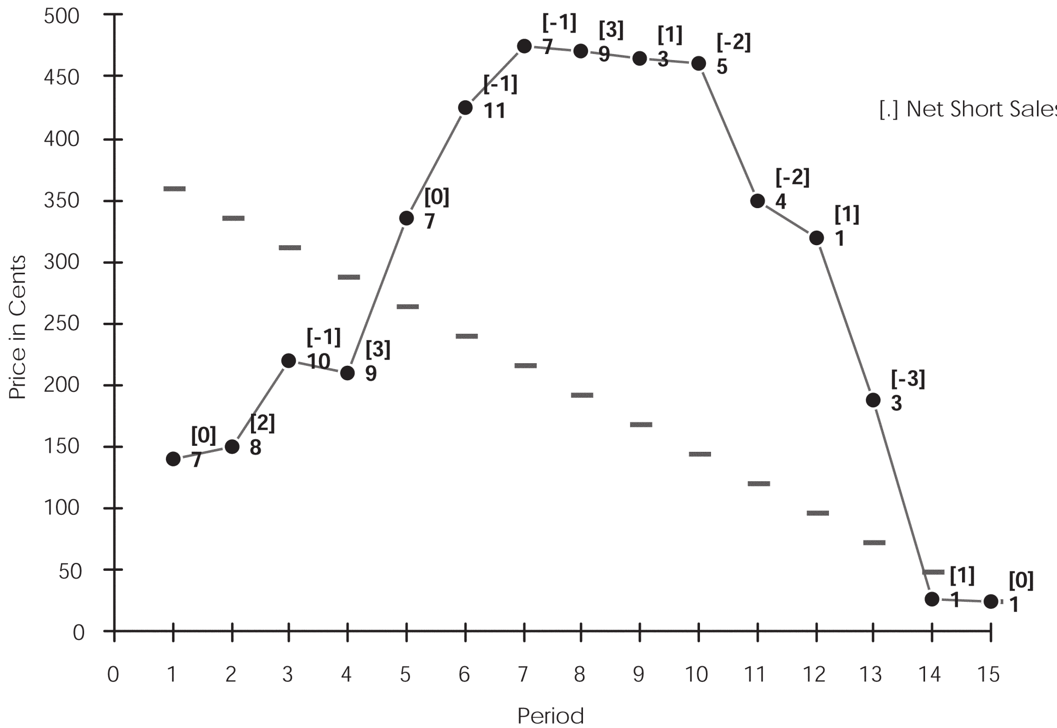
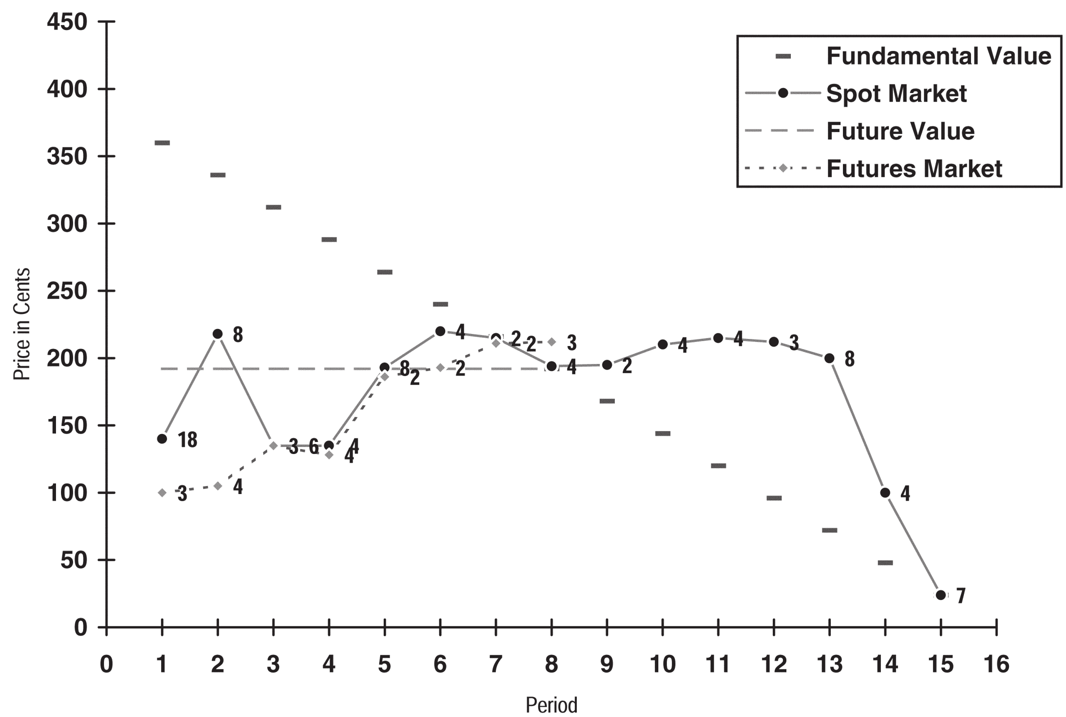


FIGURE 8
Mean Spot and Futures Contract Prices and Total Volume



Limit Price Change Rules

Many of the world's stock exchanges have imposed circuit breaker restrictions, in which trading is halted if prices move up or down by a specified amount. Arguably, the purpose of these rules is to allow traders to take stock of the current situation and to break up the formation of self-fulfilling price expectations.

To test a specific form of circuit breaker that strongly limits price "volatility," KSWV conducted a series of six experiments in which prices each period were bounded by a ceiling and floor equal to the previous period's closing price plus (or minus) twice the expected one-period dividend value. They found that these *limit price change rules do not prevent bubbles. If anything, they are more pronounced in duration*. Traders perceive a reduced downside risk, inducing them to purchase shares that increase and prolong the bubble. However, when the market breaks, it moves down by the limit and finds no buyers. Trading volume is zero in each period of the crash as the market declines by the limit each period (see Figure 9).

Call Markets

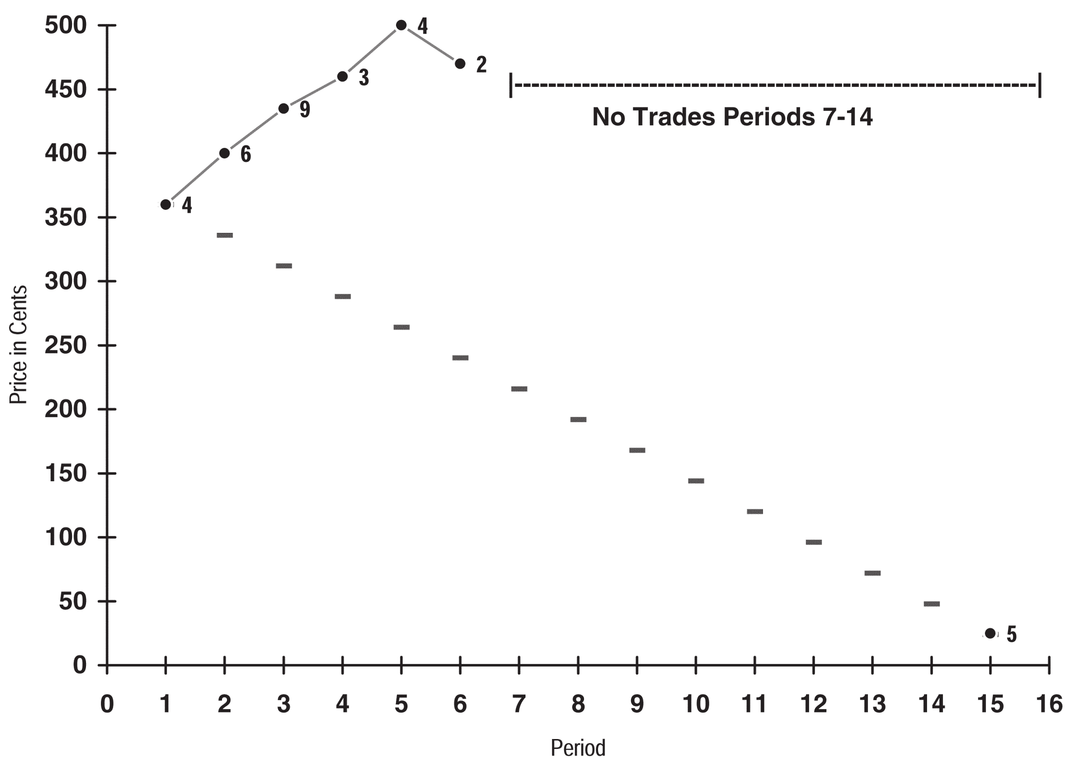
The trading institution used in the studies was a continuous double auction (CDA). The CDA is the standard mechanism used on most stock exchanges, namely, the bid-ask improvement rule, with trades occurring when agents accept the standing bid or ask.

Van Boening, Williams, and LaMaster (VWL) [1993] test whether executing all of the fifteen periods of trading at once, via a call market for each, would tend to aggregate information, eliminating intraperiod price trends and damping expectations of capital gains. In particular, VWL use a *uniform price call market,* in which bids and asks are simultaneously submitted to the market and a single market clearing price is determined where the bid and ask arrays cross. That is, buy orders are limit orders specifying the maximum price and quantity that a buyer is willing to trade, and sell orders are limit orders specifying the minimum price and quantity a seller is willing to trade. All the sell and buy orders are gathered in one book in which a *single price* is found where exchanges occur so that buyers pay less than or equal to what they bid, and sellers receive payment greater than or equal to their ask. VWL found that *the price patterns in their call market institution are consistent with the patterns found in the continuous double auction asset market*. Thus, switching from a continuous bid-ask spread market to a call market does not affect the characteristics of price bubbles.

Mathematical Modeling of Momentum and Overreactions

Current theory offers no systematic insight into the above experimental data. Nor does it illuminate the problems and issues that confront practical securities

FIGURE 9
Mean Contract Price and Total Volume: Limit Price Change Rule



trading and marketing. In particular, the prolonged deviations from fundamental value (or dividend value in the experiments), large price movements in the absence of significant news, and sudden unexpected crashes are puzzling in terms of classical theories. These theories assume unlimited arbitrage capital and unbounded rationality, which would restore prices to realistic value before deviations became large. In 1998, both of these assumptions were severely strained by the demise of Long Term Capital Management, whose primary investment strategy was to conduct option arbitrage based on the Black-Scholes model (see Frantz and Truell [1998]).

The inability of current theories to explain, even qualitatively, some key features of experiments and practical experience has led to an examination of these theories and the extent to which they need to be modified to be compatible with the observations. Classical economics focuses primarily on equilibrium phenomena. Modern theories that explain the time evolution toward equilibrium have often used differential equations, usually with a probabilistic or stochastic component. A central assumption has involved the dependence of the change of price on the deviation of the price from fundamental value. This precludes any overshooting of the price through that fundamental value, and is therefore incapable of describing overreactions and oscillations in the market (except through random or stochastic factors). In mathematical terms, the role of price change history (or in trading terminology, the trend) is neglected.

A second concept key to the traditional theories is the assumption of infinite capital that is available to eliminate market inefficiencies. In practice (as in the experiments), the pool of money available is limited. Underwriters, for example, are keenly aware that if they bring too much supply to the marketplace, the price of the asset will suffer, even though the valuation may be sound.

We discuss next a theoretical development based on making these two important changes (i.e., incorporating price trend and the finiteness of cash and assets) within a differential equations framework that relates the change in price to the underlying microeconomic motivations for buying and selling, similarly to the modern theories of price adjustment.

In addition to these issues, a large body of research, known as technical analysis, attempts to identify patterns on price charts that may indicate whether a trend is likely to continue or terminate. Of course, such a possibility is ruled out by the (weak) efficient market hypothesis, which maintains that prices alone have no predictive value. Many academicians are quite skeptical of these ideas, while some practitioners use them routinely in trading and marketing securities.

Modern theories of price adjustment (see, e.g., Watson and Getz [1981]) stipulate that relative price change occurs in order to restore a balance between supply, *s*, and demand *d*, each of which depend on price:

$$\frac{d}{dt}\log p = \frac{1}{p}\frac{dp}{dt} = G\left(\frac{d(p)-s(p)}{s(p)}\right) = F\left(\frac{d(p)}{s(p)}\right) \quad (5.1)$$

where $p(t)$ is the price of a share at time t, and $d(p)/s(p) - 1$ is excess demand (normalized by supply). Equation (5.1) implies that the relative change in price depends upon a function, F, of demand and supply at that price, p. This function, F, must have the property that when supply and demand are equal there is no change in price. On the other hand, when demand exceeds supply, so that $d(p)/s(p) > 1$, prices rise, and conversely, when supply exceeds demand, so that $d(p)/s(p) < 1$, prices fall.

The larger the ratio of demand to supply, the more rapidly prices rise.

In mathematical terms, this means that the function F has the properties

$$F(1) = 0 \text{ and } F' > 0 \quad (5.2)$$

With the standard assumptions that $d(p)$ and $s(p)$ are monotonic, condition (5.2) ensures that this equilibrium point is unique.

The vast majority of the phenomena discussed previously cannot be explained on the basis of this formulation. In Caginalp and Ermentrout [1990] and Caginalp and Balenovich [1994], the basic theories were generalized by preserving as much of the foundation as possible, e.g., the structure of the price equation, while modifying some of the concepts that are in clear conflict with the experiments. Studies showed that in laboratory experiments such as PS (see Figure 3), current prices were strongly influenced by previous prices. This trend dependence would be difficult to explain with supply and demand depending on price alone.

The next step in specifying the detailed form of the equations is to determine a functional form of d and s. The comments above justify the dependence of demand and supply on price trend as well as price itself. This is expressed generally as

$$\frac{d}{dt}(\log p) = F\left(\frac{d(p, p')}{s(p, p')}\right) \quad (5.3)$$

We need to specify the dependence of demand, d, and supply, s, on the price, p, and price derivative, p'. This dependence is achieved through an investor sentiment function, ζ, that includes all motivations for purchasing the asset. The investor sentiment determines an index or flow function, k, which measures the flow from cash to the asset. One can regard k probabilistically as the likelihood that a unit of cash will be submitted for a purchase order of the asset. Consequently, k must take on values between 0 and 1. When k is close to 1, investors are eager to buy the asset, and when k is near 0, they have little interest. Since ζ can take on any value, we need a transformation from ζ to k. This is achieved through a smooth function, such as that seen in (5.5) in the Appendix. In principle, this sentiment function, ζ, can depend on a variety of factors that influence investor decisions. We focus on two factors: the price trend, ζ_1, and fundamental valuation, ζ_2.

As discussed in the Appendix, Equations (5.6´) and (5.7´) express the simplest mathematical expression for these ideas. Equation (5.6´) expresses ζ_1 as the relative price change multiplied by a factor, q_1, which indicates the weighting that the investor group places upon the trend. Equation (5.7´) stipulates that ζ_2 is a weighting factor, q_2, times the relative discount of the price from the fundamental value, $p_a(t)$. At a deeper level, we can express ζ in terms of its dependence on the price changes in the past with the more recent events weighted most strongly. This leads to (5.6). A similar delay effect in terms of recognizing undervaluation is expressed by (5.7).

Thus, Equations (5.3)-(5.7) constitute a system of differential equations (the momentum model) that can be studied computationally upon specifying the parameters such as q_1 and q_2. These constants are not known but can be evaluated experimentally for a particular investor group. Then, one can use the computer calculations for the differential equations to predict price behavior in subsequent experiments.

We derive a system of differential equations within a general class represented by (5.3) next (see the appendix for a derivation and exposition of these equations). Generally, we denote flow demand rates (a desired rate of accumulation of asset shares) and flow supply rates (a desired rate of accumulation of cash) using lower-case symbols as above in Equation (5.1), while upper-case symbols are used to denote finite supplies of shares or cash.

We describe a mathematical model that involves a closed system (i.e., a fixed number of shares of a single asset plus a fixed amount of cash). This is ideally suited for studying the asset market experiments. The mathematical system can also be generalized to incorporate influxes or outflows of cash or shares into the experiment.

We begin by stating some stock flow identities, then introduce the equations representing the behavioral assumptions that underpin the sentiments governing the supply and demand rates. We analyze a closed system containing M dollars and S shares. The demand, d, for shares is expressed as the available cash multiplied by the rate, k (normalized so that it assumes values between 0 and 1), that investors desire to accumulate shares (place purchase orders) . A similar description applies to the desire to accumulate cash (place sell orders). If we let B be the fraction of the total value of assets held in the form of shares ($1 - B$ is the fraction held in cash), then $B = pS/(pS + M)$, and the stock-flow *identities* we use can be written as:

$$d = k(1-B),\ s = (1-k)B,\ \text{and}\ \frac{d}{s} = \frac{k}{1-k}\cdot\frac{1-B}{B} \quad (5.4)$$

All the behavioral features of this system will be defined in terms of k, where k can be thought of as the velocity (turnover) of the stock of money required to express the demand for shares in this closed system. Similarly, $1 - k$ is the velocity of the stock of shares required to express the supply of shares.

To develop the behavioral hypotheses concerning market decisions, let $p_a(t)$ denote the fundamental value of a share at time t. If k depended only on $p_a(t)$, we would have a generalization of the theory of price adjustment written in terms of the *finiteness* of assets and *delay* in taking action. However, if the rate k is specified through *investor sentiments,* the desire to accumulate shares, or preference for shares over cash, then price can adjust based on investor perceptions.

To understand the dependence of investor sentiment on the history of price change, consider the motivation of an investor who owns the security as it is undervalued but still declining. The choice available to this investor is to either sell or to wait in the expectation that those with cash will see the opportunity to profit by purchasing the undervalued security. The issue of distinguishing between self-maximizing behavior and *reliance* on optimizing behavior of others is considered in the experiments of Beard and Beil [1994] on the Rosenthal conjecture [1981] that showed the unwillingness of many agents to rely on others' optimizing behavior. Not everyone assumes that others will automatically act in their immediate self-interest.

The dynamical system is closed by relating two types of investor sentiment into trading motivations. In particular, the total investor sentiment or preference function is expressed as the *sum* of the price trend and the price deviation from fundamental value.[6] In each case, the basic motivation is summed with a weighting factor that declines as elapsed time increases. In the case of the trend, this means that recent price changes have a larger influence than older ones. For the fundamental component, it means that there is some lag time between an undervaluation and investor action. The weighting factors are assumed to be exponentials, so that there is a gradual decline in the influence of a particular event.

In general, when price is below fundamentals, value investors start buying shares, thereby moving the price higher. This provides a signal that draws trend-based buyers into the market, precipitating a further increase in the rate of price change, which further fuels the price increases. As prices rise above fundamentals, value investors start to sell, increasing their liquidity and reducing the liquidity of trend-based investors. As the trend reverses, the momentum traders continue the sell-off until prices drop to (or below) fundamental value.

The numerical computations of the model confirm that if the trend-based coefficient is sufficiently small, the price evolves rapidly toward $p_a(t)$ with little or no oscillation. This corresponds to a classical rational expectation model (see Tirole [1982]). If trend-based motivations are increased further, the price oscillations increase in magnitude and frequency. As the trend-based motivations are increased, they reach a point where the price oscillations become unstable in the sense that they increase in magnitude without bound. Behavior in this model is reflected in an increased (or decreased) desire to accumulate shares, but of course it is impossible for the market as a whole to acquire more shares, the quantity of which is fixed. So autonomous changes in the desire to accumulate shares alter the price by precisely the amount required to induce a desire in the market as a whole to hold the existing stock. But the relative holdings of that stock by different types of investors will change over time as part of the equilibrating process. The equilibrium is not that of rational expectations theory unless there are no momentum traders and all investors are motivated by fundamentals.

What is the extent to which these equations can predict the price evolution in experiments? In principle, once one knows the dividend structure and the trading price of period 1, the rest of the trading prices can be predicted if the parameters have already been estimated from previous experiments. This out-of-sample prediction approach has been implemented and compared with other methods (see Caginalp, Porter, and Smith [1999]), and is discussed in Section VIII.

Applying Principles From the Laboratory to Field Data

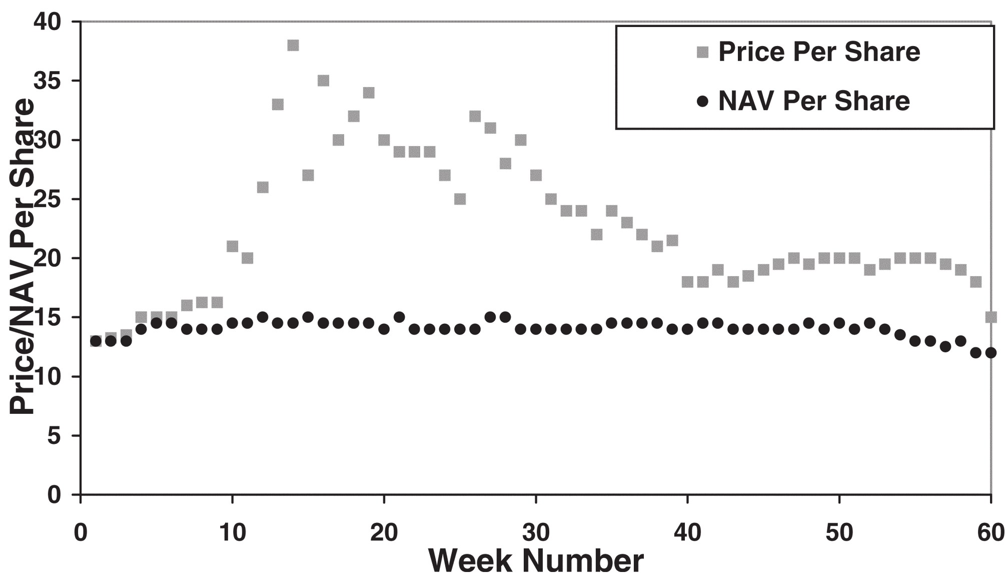
The experiments described above provide several parallels that researchers have used in analyzing data found in field stock markets. The structure of the experiments is similar to that of a closed-end fund, in which investors can find the net asset value (NAV) of the fund published in most financial newspapers.[7] Consider the data in Figure 10, which lists the average weekly share price and corresponding NAV for the *Spain Fund*. The price of the Spain Fund shares from July 1989 to August 1990 begins at a discount from NAV and rises to a premium of 250% over NAV by week 15, and ultimately “crashes” back to a discount by week 61.[8]

Trader Experience

One of the replicable results from the experiments described earlier is that once a group experiences a bubble and crash over two experiments, and then returns for a third experiment, trading departs little from fundamental value.

Renshaw [1988], taking his cue from this result, hypothesizes that the severity of price bubbles and crashes in the economy is related to “inexperience.” As time passes, new investors enter the market, old investors exit, and the proportion of investors remembering the last stock market decline changes. He examined the relationship between major declines in the Standard & Poor’s index and the length of time between major declines. The time between crashes is his proxy for investor inexperience. An OLS regression of the measured

FIGURE 10
Share Price and NAV: The Spain Fund 6/30/89–8/24/90



extent of the index's decline, Y, on the time since the previous decline, X, yields the estimate:

$$Y = 5.5 + 0.90X;\ R^2 = 0.98$$
$$(t = 15.1)$$

The greater the magnitude of a crash in prices, the longer it will be before its memory fades and we observe the termination of a new bubble-crash cycle. This analysis, unlike the replication of laboratory experiments, cannot distinguish between events that are spaced far apart in time because they are rare events, and the causal effect hypothesized by the regression. Hence, this relationship may be suspect.

Time Series Methods as Link Between the Laboratory and the Field

Differential equations are a powerful modeling tool, because they incorporate specific postulated forms of behavior and impose physical constraints like the conservation of cash and shares. On the other hand, the assumptions used to derive the equations may be controversial. We address this point by applying non-parametric statistical tests of the predictions of the differential equations to data from experiments in which we control variables such as dividend value and the inventory of cash and shares.

Another approach to modeling is standard time series analysis, which addresses two key questions. First, can one identify momentum and the extent to which it influences price movements in world markets, and then use this information to make out-of-sample predictions? Second, can one use these procedures to make a quantitative link between the phenomena observed in laboratory experiments and world market data?

A very simple model for understanding asset prices is the *random walk* model, which relates the price at time t, denoted $y(t)$, to the price one time unit ago in the following way:

$$y(t) = y(t-1) + w(t) \tag{6.1}$$

where $w(t)$ is a sequence of independent random disturbances with zero means and equal variances (see Shumway [1988, p. 129]). This is the simplest of the Box-Jenkins or ARIMA models, which can be summarized as follows.

The basic ARIMA models involve components that are autoregressive (AR), meaning they link the present observation components $y(t)$ with those up to h times earlier, $\{y(t-1),\ldots,y(t-h)\}$, and the moving averages (MA) of the error terms experienced in the previous q members of the time series, $\{\varepsilon(t-1), \ldots, \varepsilon(t-q)\}$. The observations, $y(t)$, can be differenced (denoted Δ) so that if the original series is non-stationary, the methods are applied to the sequence $w(t) := \Delta y(t)$, which is the sequence $\{y(t)\}$ differenced v times. The general (ARIMA(h, v, q) model can then be written as:

$$w(t) = \phi_1 w(t-1) + \phi_2 w(t-2) + \ldots + \phi_v w(t-h) + \varepsilon(t) + \theta_0 - \theta_1 - \ldots - \theta_q \varepsilon(t-q) \tag{6.2}$$

in terms of the coefficients or "process parameters" ϕ and θ. In particular, ARIMA (0, 1, 0), i.e., $h = 0, v = 1, q = 0$, is just ordinary random walk, while ARIMA (0, 1, 1) is simply an exponential smoothing scheme.

Analyzing market data is generally difficult, as it is influenced by many random unknown changes in fundamentals. To control for this, Caginalp and Constantine [1995] used data on two closed-end funds, the Future Germany Fund (FGF) and the Germany Fund (GF), consisting of the *same portfolio and having the same manager*. Closed-end funds trade like ordinary stocks, and may have a premium or discount to the net asset value (NAV). The fundamental changes are identical for the two stocks so that the ratio of the price of the two funds should not change during the time period. They define

$$y(t) = \frac{\text{Price of } FGF_t}{\text{Price of } GF_t} \tag{6.3}$$

and analyzed the time series of closing prices from the inception of the latter fund (FGF) until the 1,149th day. The efficient market hypothesis, which would predict that $y(t)$ would fluctuate randomly around a value of unity, was tested using the "sign test" and the "turning point test" (see, for example, Krishnaiah and Sen [1984]).

For the entire data, the number of runs deviated by 29 standard deviations from that expected from the null hypothesis of constant value plus noise. Similarly, the turning point test deviated by 6.8 standard deviations from the null hypothesis. The standard deviation of a data set is a measure of its dispersion, so a large standard deviation corresponds to a high probability that a particular measurement will fall far from its mean, while a small standard deviation means it will likely be close to its mean. If a set of measurements deviates by two or more standard deviations from the mean, it is very unlikely to be a result of randomness.

Once the null hypothesis has been rejected, the Box-Jenkins procedure can be applied. Applying this procedure to the entire data, they found that $v = 1$ is necessary and sufficient. Examination of the autocorrelation function resulted in $h = 1$ and $q = 1$, as the correlations drop dramatically in the next order. The emergence of a particular ARIMA model, i.e., (1, 1, 1), rather than the (0, 1, 0) associated with random walk, further confirmed the existence of trend-based (momentum) components in the data. The ARIMA model selected by the data using this procedure was found to be

$$y(t)-y(t)=0.5\{y(t-1)-y(t-2)\}+\varepsilon(t)+0.8\varepsilon(t-1) \tag{6.4}$$

The coefficients 0.5 and 0.8 are 9.6 and 21.6 standard deviations, respectively, away from null hypothesis values of 0, in which yesterday's price is the best predictor of today's price.

Consequently, the concept of a lagged difference structure emerged quite naturally from the data, as Equation (6.4) is a relation between today's rate of change, $y(t) - y(t-1)$, compared with yesterday's, $y(t-1) - y(t-2)$. Thus, the ARIMA procedure leads to the conclusion that the best predictor of prices is very far from a random perturbation from yesterday's price.

The results suggest a very basic motivation in trading. In the absence of any change in fundamental value, there are two simple views possible about price movement: 1) that price will be essentially unchanged from the day before, and 2) that today's price *change* will be essentially unchanged from yesterday's. The coefficient 0.5 in Equation (6.4) effectively interpolates between the two strategies and indicates that the investors who generated this data set were equally inclined to be influenced by yesterday's price change as they were by the price itself.

In financial forecasting, the use of "out-of-sample" predictions is a valuable test to ensure that there is no "overfitting" of the data. Caginalp and Constantine [1995] used the first-quarter data (the first sixty-four days) to predict this quotient during the next ten days without updating the coefficients. The actual values for days 65 to 74 were well within the 95% confidence regions.

In a more extensive test, they also used the ARIMA (1, 1, 1) model to forecast with updated coefficients by using the first N days in order to estimate the coefficients and to forecast the $(N+1)^{th}$ day's quotient. Beginning with the first sixty-four days, they predicted the next quarter's price quotients on a day-by-day basis. The predictions were again within the 95% confidence intervals and were better than both the random walk prediction and the constant ratio (efficient market) predictions by three standard deviations, as measured by the Wilcoxon matched pairs signed ranks test or binomial distribution comparisons.

Such statistical methods can potentially establish a quantitative link between the laboratory experiments and the world markets. Toward this end, the ARIMA model and coefficients constructed using the first quarter of the FGF/GER data are used to forecast the experiments done by PS. In the set of experiments considered, the participants traded a financial instrument that pays 24 cents during each of fifteen periods. Hence, the fundamental values of the instrument are given by

$$P_a(t)=3.60-.24t \tag{6.5}$$

We let $P(t)$ denote the experimental values of price, $P_a(t)$ the fundamental value determined by summing the expected dividends at that time, and define

$$x(t):=P(t)/P_a(t) \tag{6.6}$$

We can apply the same time series framework used for $y(t) = FGF/GER$. The time series $y(t)$ and $x(t)$ both possess the key property that the temporal changes in their fundamental value have been eliminated. Consequently, the efficient market hypothesis predicts the same value (in time) for both.

We would like to examine the extent to which data from world markets can be used to predict experiments and vice versa. If this can be done successfully, it would provide considerable evidence that the mechanism underlying price dynamics is similar in both cases. It would also lend support to the concept of searching for microeconomic mechanisms for price change in the absence of fundamental changes in valuation.

Toward this end, Caginalp and Constantine [1995] used the ARIMA(1, 1, 1) model with the coefficients obtained from the FGF/GER data to make predictions on the experiments. These predictions were then compared with the null hypothesis, namely, that $x(t) = 1$ for all t.

The Wilcoxon paired difference test confirms that the ARIMA(1, 1, 1) with the original coefficients allows rejection of the null hypothesis that $x(t) = 1$, with a statistical significance of $p = 0.007$. In other words, the probability that the ARIMA model's superior predictions are attributable to chance is less than 1%. This result is remarkable because not only the model, but the coefficients as well, have been determined entirely from New York Stock Exchange data.

We believe this is a promising direction for future research in that it allows us to interpret quantitatively the results of experiments in terms of world markets, and vice versa. It offers the possibility that one can use experiments to make statements that go beyond qualitative conclusions and to examine the extent of that particular mechanism, because price dynamics are universal across different investor populations.

Testing the Momentum Model: Experiments

If the parameters of the system of differential equations were estimated, it would be possible to predict the entire price path if we knew the opening price $P(0)$. Thus, if one could control the opening period price, this model would predict the entire price path. This is the motivation for the opening period price control experiments we discuss next.

Using Price Controls to Initialize the Model

Caginalp, Porter, and Smith (CPS1) [1999] estimated, using ordinary least squares, the two parameters representing the strength of trend-based (q_1) and fundamental value-based (q_2) investing from a set of baseline experiments in which the opening price is unconstrained. These parameters were then used to determine the price predictions when the opening price is restricted to trade in a specified range. We used price controls so that we could replicate experiments with identical opening prices, i.e., we control for the opening price. The price controls were always below the initial $3.60 expected value, and ranged from a control interval of [$1.40, $1.60] to [$2.90, $3.10].

Two types of price control experiments were conducted. The first set used the standard $0.24 dividend, while the second set doubled the dividend distribution ($0.48 in experiment money) and cash, but made the conversion rate of experiment money into U.S. currency one-half, so that there would be no difference in real money space. Figure 11 provides the momentum model predictions under the different price control treatments.

Figures 12 and 13 show the ratio $\Phi_t = P_a(t)/P_m(t)$, where $P_a(t)$ is the actual mean contract price in period t and $P_m(t)$ is the prediction from the momentum model, for each of the opening price control treatments. Thus if $\Phi_t = 1$, there is a perfect match between the actual mean price and the momentum prediction for period t; if $\Phi_t > 1$, the momentum model prediction is less than the actual price for period t; if $\Phi_t < 1$, the momentum model prediction is greater than the actual price for period t. The data suggest that *the momentum model underestimates the mean contract price in the early trading periods and overestimates the price in the later trading periods. The data appear to imply that there is an asymmetry between the bull and bear phases of a bubble that is not captured by the model.*

While the momentum model is not predictive within 5%, it does have some qualitative properties that can be exploited. Recall that in the momentum model, when prices are below $P_a(t)$, there is a tendency for buy orders to increase due to the expected return. As prices approach fundamental value, the momentum is higher due to increasing prices. Thus, if we consider positive price differences from one trading period to the next, the momentum model would predict that the sum of these differences would be greatest when the initial undervaluation is greatest. The following regression was estimated:

FIGURE 11
Momentum Model Predictions for Various Initial Price Controls

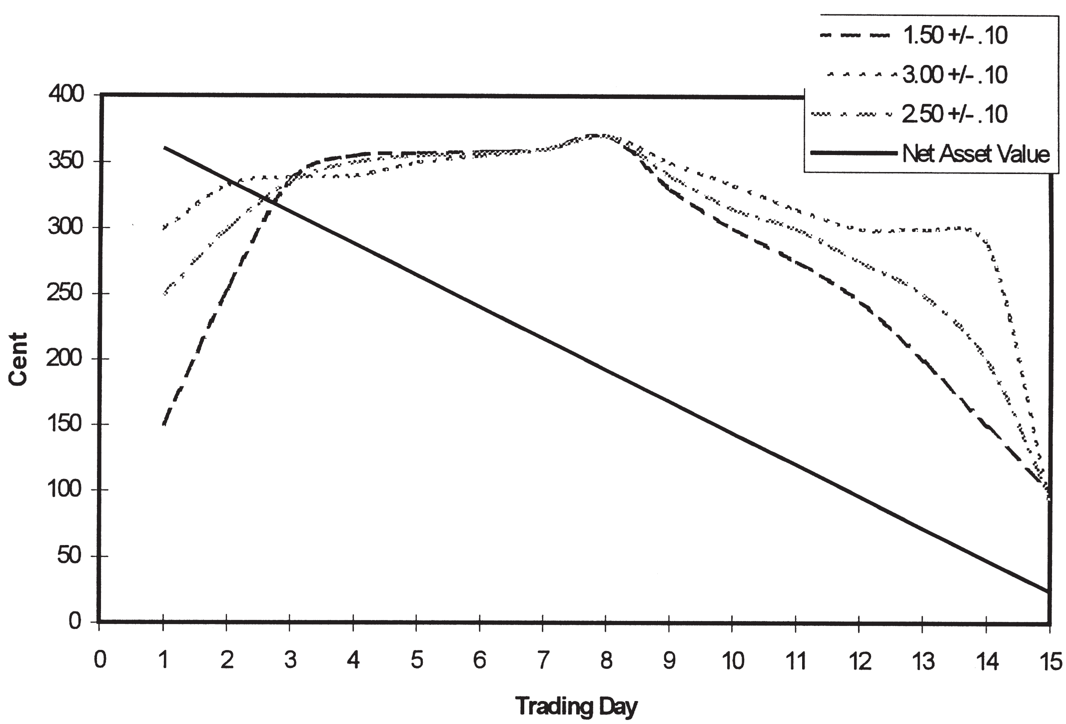


FIGURE 12
Actual Prices/Momentum Predicted Price ($0.24 Dividend Case)

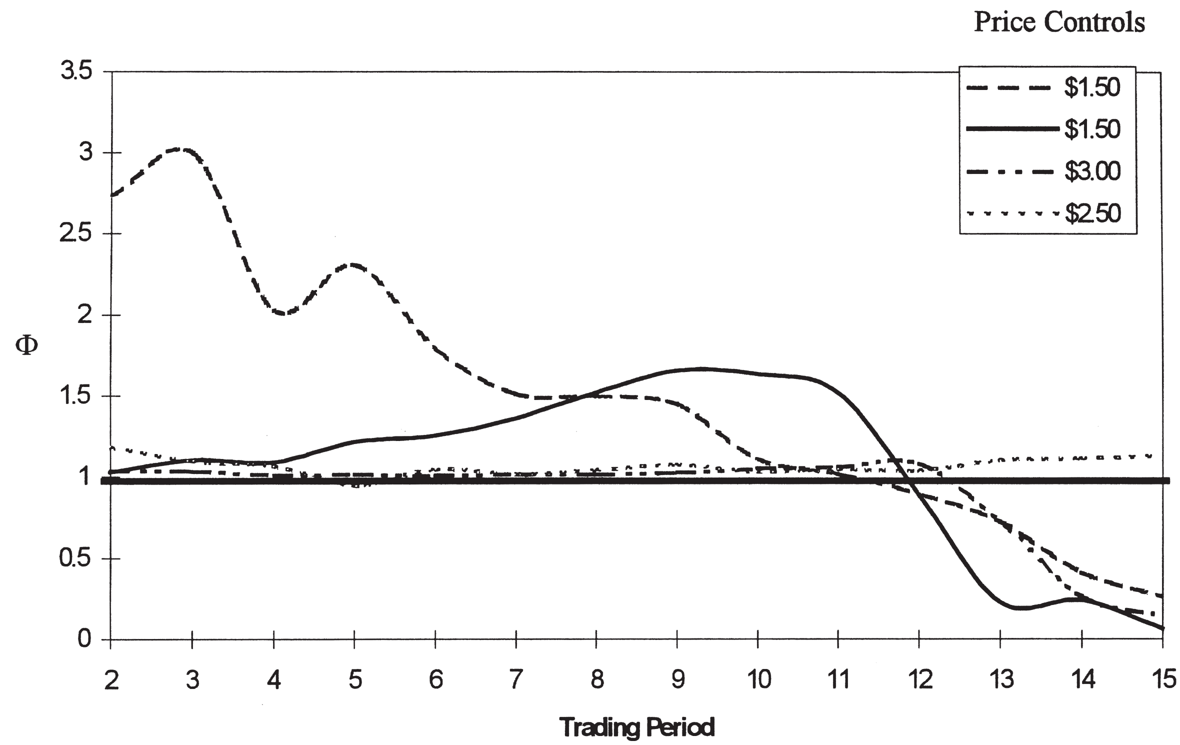
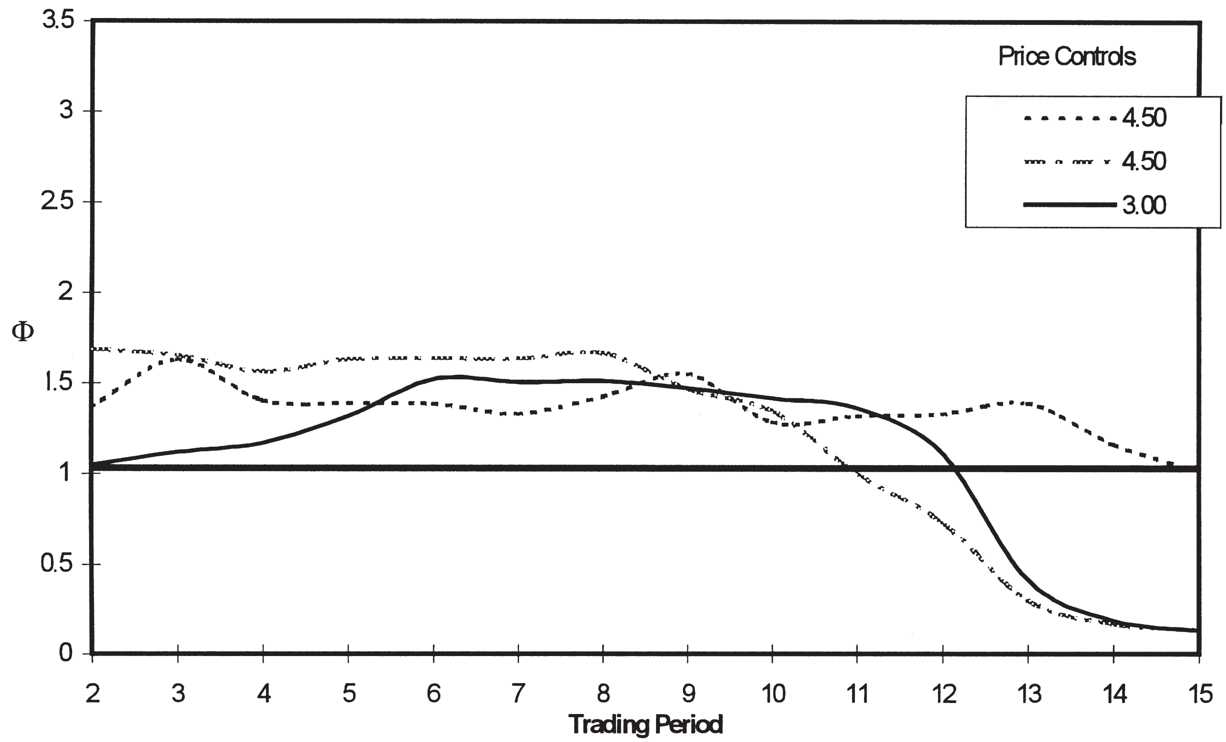


FIGURE 13
Actual Prices/Momentum Predicted Price ($0.48 Dividend Case)



$$\text{Sum}_i = 0.9 + 0.9\ (\text{undervaluation})_i$$
$$(s.e. = 0.61)$$
$$(p\ \text{value} = 0.21)$$

where i indexes the experiment. The prediction is that β > 0. CPS1 *find a larger initial undervaluation produces a larger positive price movement.*

It is not surprising that the momentum model did *not* accurately predict the entire price path, because the momentum model predicts fifteen periods in advance and is independent of the characteristics of the group that is trading. Updating, based on current and past trading activity, would provide a better calibrated model. In CPS1, we use the previous $j - 1$ experiments to obtain optimal values of the investor sentiment parameters for each of the experiments and average these values to get new parameter estimates. This updated calibration method allows the parameters to adjust dependent on the most recent information from the market.

Price Forecasting Models

While the original momentum model does not have high predictive powers for large times, it may have better predictive power relative to other price forecasting models. In CPS1, the momentum model was pitted against the following forecasting methods to determine which method predicts best.[9]

Expert trader model. CPS1 recruited the most profitable traders from several baseline bubble experiments to be "professional" forecasters in later experiments.[10] These professional forecasters derived their entire earnings from the accuracy of their price forecasts. These subjects did not trade in the market but could see all bids, asks, and contract prices in the market. In addition, they would submit price forecasts for the asset for the next two periods. They were paid for each of their forecasts as follows.

Let C_t be the price forecast, F_t the fundamental value, and P_t the actual mean contract price in period t. Then the reward for the period t forecast is:

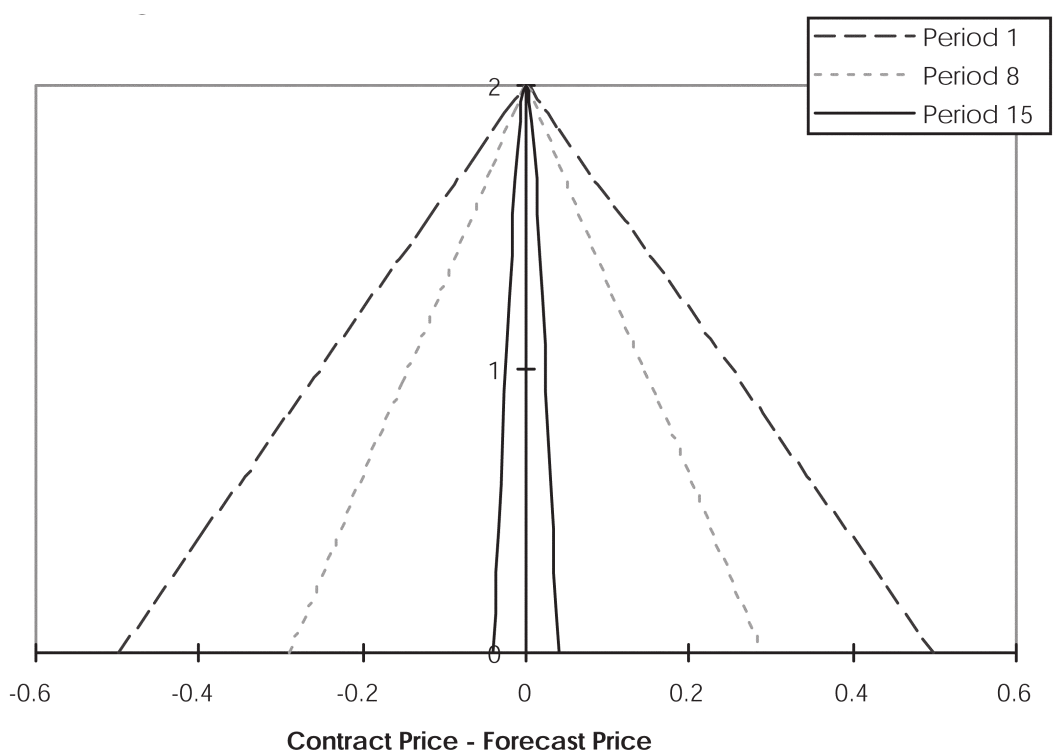
$$\text{Maximum}\begin{Bmatrix} 0 & \text{or} \\ \$2.00\left(1 - \left(\left|P_t - C_t\right|\right) / \left(0.15\left[F_t\right]\right)\right) & \end{Bmatrix}$$

Figure 14 shows the payoffs for the forecasters as a function of the period in which the forecast was made and the deviation from the actual contract price for the period.

ARIMA (1, 1, 1) and (0, 1, 0). The random walk model [ARIMA (0, 1, 0)] and the ARIMA (1, 1, 1) model are standard time series forecasting models, and provide a simple baseline method to determine the efficiency of other forecasting methods.

Excess bids model. SSW proposed a price adjustment model where price changes are a linear func-

FIGURE 14
Forecaster Incentive Function



tion of the excess bids (bids tendered over asks tendered). The excess bid variable provides a proxy for the excess demand for the asset. This is similar to a Walrasian price adjustment model, which stipulates that price responds in the direction of the excess demand for the asset. SSW use the difference between the number of bids and number of asks submitted as a proxy for excess demand. In particular, they estimate the following ordinary least squares model:[11]

$$\bar{P}_t - \bar{P}_{t-1} = \alpha + \beta\left(B_{t-1} - O_{t-1}\right) + \varepsilon_t$$

where $\bar{P}_t$ is the mean price in period t, α is minus the one-period expected dividend value (adjusted for any risk aversion), β is adjustment speed, B_{t-1} is the number of bids to buy tendered in period $t-1$, and O_{t-1} is the number of offers to sell tendered in period $t-1$. Price change in this model has three components: 1) the risk-adjusted per period expected dividend payout, 2) an increase (decrease) due to excess demand arising from homegrown capital gains (losses) expectations (a Walrasian measure of which is excess bids, $B_{t-1} - O_{t-1}$), and 3) unexplained noise, ε_t . Forecasting in the experiments uses a least squares regression that is updated across all previous experiments to obtain the parameter estimate $\hat{\alpha}$ and $\hat{\beta}$. In particular, the forecasted price in period $t + 1$ is given by:

$$P_{t+1} = P_t + \hat{\alpha} + \hat{\beta}\left(B_{t-1} - O_{t-1}\right)$$

The results of the forecasting experiments show:

(i) *The momentum model and professional forecasters' price predictions have similar absolute errors.*
(ii) *The momentum model has superior two-period ahead forecasts relative to the other forecasting models.*
(iii) *The ARIMA models are relatively the worst forecast models.*
(iv) *The professional forecasters update price predictions based on a forecast surprise.*
(v) *The excess bids model has a slightly better one-period forecast than the momentum model.*

This last result suggests that it may be desirable to incorporate other market information into the momentum model in order to update the price path. In particular, the momentum model could incorporate excess bids and forecast surprises into its framework along with market conditions (cash and share holdings in the market).

Liquidity and Price Formation

Rational expectations theory proposes a unique value for a financial instrument that reflects all information among the participants as to its worth. Any temporary moves away from this value will be quickly arbitraged. However, it is generally recognized by investment houses and traders that a large supply of stock in the marketplace has a significant and perhaps lasting effect on stock prices. Portfolio managers routinely submit large orders slowly over time and through various brokers so that they will not cause large movements of a stock's price against them as they seek to change their position. Thus, there is a split between theory and the beliefs that drive practice. Practitioners constantly talk about market "liquidity" as an indication of the ability of stocks to absorb pulses in the order flow or to maintain a price level or trend.

In this section, we introduce a notion of liquidity that can be interpreted in terms of the momentum model, and used as a treatment in designing experiments.

Modeling Liquidity Value

Both the experimental evidence and computations based on the differential equations suggest that the amount of available cash in relation to shares is a potentially important factor in price movement.[12] Within the realm of rational expectations, there is no mechanism for an excess of cash or of the asset to induce deviations from fundamental value. Any addition to cash balances will be held in idle accounts if the available shares are priced at their fundamental value. But this is not generally true for the momentum model, where an increased supply of cash may increase purchases of shares by momentum traders who would like to buy in proportion to the percentage increase in price, but are cash-constrained. We saw in section IVB that when we allowed margin purchases there was a significant increase in the amplitude of bubbles. The momentum model predicts this will occur if any momentum trader's purchases are constrained by his cash position. Borrowing allows that constraint to be loosened.

Caginalp and Balenovich [1999] reinterpret the original differential equations to define and analyze liquidity in a precise manner, as follows. Consider a closed market containing S shares of an income-generating asset, and M dollars distributed arbitrarily among participants at the outset and subject to change over time.

Let the price of the single asset be denoted again by $p(t)$ and define liquidity as the ratio of cash to shares, $L=M/S$, which is measured in dollars per share. Given that the relative price changes linearly with excess demand we have:

$$\frac{\tau_0}{p}\frac{dp}{dt} = \frac{d}{s} - 1 \tag{8.1}$$

for a time scale τ_0. Thus, from the definitions in (5.4), we find

$$B = \frac{S \cdot p}{S \cdot p + M},\ 1 - B = \frac{M}{S \cdot p + M} \tag{8.2}$$

$$\frac{B}{1-B} = \frac{S \cdot p}{M} = \frac{p}{L} \tag{8.3}$$

so that $B^{-1}(1-B)p = L$ is time-invariant. Equation (8.3) leads to the derivative identity

$$\frac{dB}{dt} = B(1-B)\frac{1}{p}\frac{dp}{dt} \tag{8.4}$$

Note that there is no need for a time scale in this equation. Given the two types of investor sentiment as described in section V, and the assumption that the transition rate k is a weighted sum of the current derivative and the valuation discount, we obtain:

$$\frac{k}{1-k} = 1 + 2\frac{q_1 \tau_0}{p}\frac{dp}{dt} + 2q_2\left(1 - \frac{p(t)}{p_a(t)}\right) \tag{8.5}$$

where q_1 is the coefficient representing trend-based motivations and q_2 is the coefficient representing fundamental value motivations.

Using (5.4) and (8.3) in the price equation (8.1) results in

$$\frac{\tau_0}{p}\frac{dp}{dt} = \frac{k}{(1-k)}\frac{L}{p} - 1 \tag{8.6}$$

Note that the liquidity value, L, which represents the nominal value of all money in the system divided by the total number of shares, is a fundamental scale for price. In particular, p/L is the ratio of asset value to liquidity value. Thus, for any given p, as L increases the rate of price change increases.

Experiments Testing Liquidity

This model shows that the liquidity value obtained by dividing the total cash available by the total number of shares is a significant counterpart to the fundamental value. We test this hypothesis by conducting experiments with a spectrum of liquidity values (cash to stock ratio) ranging from \$1.80 to \$7.20, i.e., half as much cash as stock value to twice as much cash to stock value, in an environment with stock dividend value \$3.60.

A series of twelve experiments used a sealed bid-offer (SBO) one-price clearing mechanism in each trading period (see Van Boening [1991] for auction methods). At the end of the fifteenth and final period, a single payout with expectation value of \$3.60 is realized (25% probability each for \$2.60 and \$4.60; 50% probability of \$3.60). We used a payout at the last period to keep liquidity constant during the experiment. The traders, all first-time participants in an asset market study, were informed of the expected dividend at the start of the experiment.

The experiments differ only in liquidity, L, defined to be the (total) initial cash distributed to all participants divided by the total number of shares distributed. Thus, an experiment for which liquidity is $L = \$7.20$ begins with twice as much cash as stock value. Using the terminology cash-rich and asset-rich for $L > \$3.60$ and $L < \$3.60$, respectively, we find, as in earlier experiments (see Caginalp, Porter, and Smith [1998]), that cash-rich experiments result in a higher mean price. A Mann-Whitney test shows that the central tendencies of cash-rich experiments (*median* = *\$3.73*, *mean* = *\$3.75*) are higher than those of the asset-rich (*median* =*\$2.90*, *mean* = *\$2.83*), which is statistically significant at 0.0081. The two-sample *T*-test for means results in an even stronger statistical significance of 0.0007.

Examining a particular period of all experiments, the correlation between price, $P(t)$, and liquidity, L, across all twelve experiments in period t is given by

Period (t)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
.51	.69	.80	.80	.80	.70	.51	.42	.34	.63	.54	.64	.37	.67	.44

Correlation

The data indicate that the influence of liquidity is strongest in the first four periods, during which shares and cash move into different participant accounts, and that it diminishes gradually as the experiment nears the end. To study this, we divide the periods into early (periods 1–4), middle (periods 5–11), and late (periods 12–15). A linear regression of price on liquidity results in the following for the three time intervals, respectively:

$$P(t) = 1.92 + 0.289L \tag{9.1a}$$

$$P(t) = 2.74 + 0.194L \tag{9.1b}$$

$$P(t) = 2.84 + 0.146L \tag{9.1c}$$

Thus, an increase of \$1 per share of extra cash in the market means there is a 29 cent increase in the average price per share during the first four periods. Near the end of the experiment, the effect is reduced to about half, at 15 cents per dollar of liquidity, but remains significant. The diminishing role of liquidity is superseded by the fundamental value (\$3.60) and culminates in a higher constant in the later periods.

A more subtle issue is the extent to which price changes can be predicted. In particular, does either of the quantities

$$\Delta L = L - P(t-1) \tag{9.2a}$$

$$\Delta P(t-1) = P(t-1) - P(t-2) \tag{9.2b}$$

predict the next price change, $\Delta P(t) = P(t) - P(t\text{-}1)$ during any of these time intervals? As the regressions above, and Table 1 and Figure 1 indicate, the key price movements are in the early periods. Performing separate linear regressions for ΔL and $\Delta P(t\text{-}1)$, we obtain in the early periods (1–4) the results:

$$\Delta P(t) = 0.115 + 0.0797\Delta L \tag{9.3a}$$

$$\Delta P(t) = 0.0694 + 0.263\Delta P(t-1) \tag{9.3b}$$

Thus, in the early trading periods, momentum and liquidity have positive effects on price movements, but these positive effects cease to be statistically significant in later periods.

We use both statistical and differential equations models to understand the underlying mechanism for these observations. We list for each time period of each experiment the price, the previous two-period prices, and the liquidity value. Sorting the periods, we perform a regression on each period separately and estimate the following statistical model:

$$\Delta P(t) = A_1(t-1) + A_2(t-1)\Delta P(t-1) + A_3(t-1)\Delta L(t-1) \tag{9.4}$$

Hence, at time *t-1* the derivative of the price, $\Delta P(t\text{-}1)$, the deviation from liquidity value ΔL and the deviation from fundamental value (emerging through the constant, A_1) will be used to predict the next time change. Consequently, for any arbitrary initial price, *P(0)*, and liquidity value, *L*, the formula (4) allows us to compute successively from time *t-1* to *t* through period 15. This enables examination of the expected price evolution of a range of experiments with different *L* and *P(0)*.

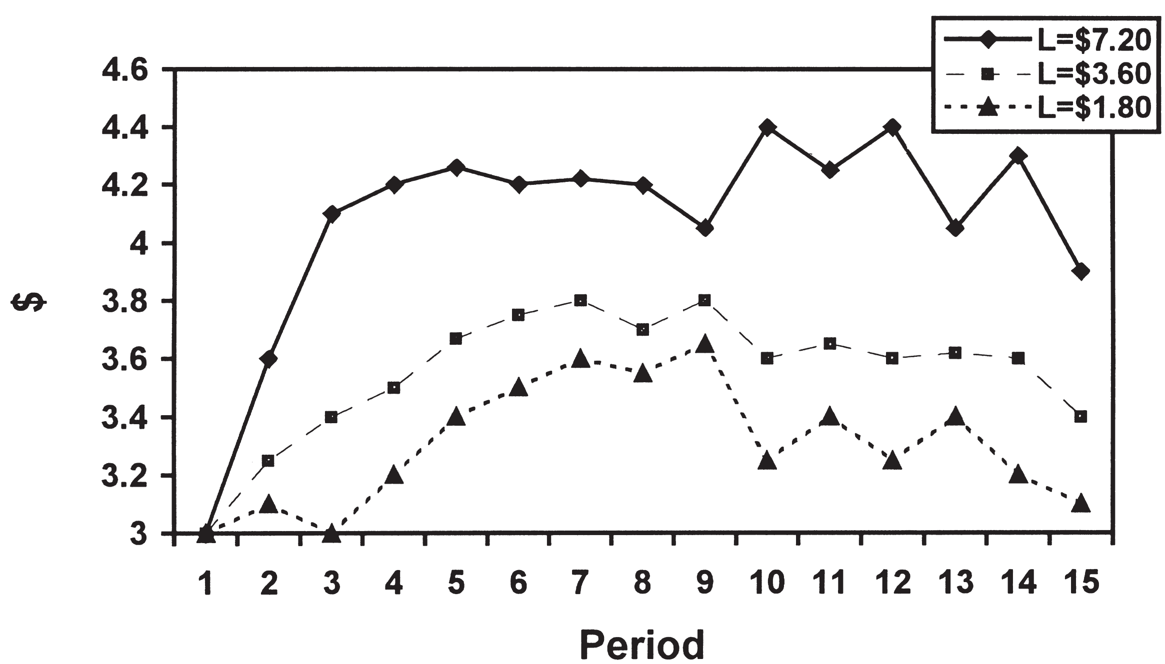
Figure 15 displays the evolution of three such cases with the same initial conditions, *P(0)* = \$3.00, with *L* = \$1.80, \$3.60, and \$7.20, respectively. Each graph displays a large price derivative initially, but the low level of liquidity for *L* = \$1.80 ultimately brings down the price in the later periods. For *L* = \$7.20, the price remains close to its peak in the late periods, and the situation is somewhere in between for the ''neutral'' case of *L* = \$3.60.

Conclusion

To understand the price dynamics in asset markets, we have summarized the results of over 150 experiments in which inexperienced traders generate a bubble relative to fundamental value and a subsequent crash back to fundamental value by the end of the market horizon. The general results are quite robust:

1. Bubbles are reduced with subject experience and academic sophistication. They are eliminated with undergraduate subjects by their third sessions, and with graduate students in their first (inexperienced) session.

FIGURE 15
Statistically Predicted Time Evolution



2. Futures markets, dividend certainty, and low liquidity tend to dampen the bubble.
3. Margin buying and limit price change rules tend to exacerbate the bubble.
4. All other treatments examined (e.g., short-selling, capital gains taxes, brokerage fees, and call markets) were neutral in their effect on the bubble.

Current finance theory offers no systematic insight into the experimental data we report. Thus, we create a dynamical system to model momentum and overreactions. The model results in a system of differential equations that allows for a wide variety of possible price patterns based on the relative strengths of fundamental-based trade and trend-based motivations. The basic feature of the model is that when price is below fundamentals, value investors start buying shares, thus causing a positive rate of change in stock prices. This signals trend-based buyers to enter the market, precipitating a further increase in the rate of price change, which further fuels the price increases. As prices rise above fundamentals, value investors start to sell, increasing their liquidity (cash balances) and reducing the liquidity of trend-based investors. As the trend reverses, the momentum traders continue the sell-off until prices drop to fundamental value. The prices may even drop below fundamental value due to selling by momentum traders, in which case the stage is set for a recovery, as value investors again start to buy.

Using data from experiments, the parameters of the model are calibrated and then tested in a new set of experiments. The main result is that *the momentum model underestimates the mean contract price in the early trading periods and overestimates it in the later trading periods.* We then look at the effectiveness of the momentum model relative to other forms of price forecasting (time series analysis, expert forecasters, and excess bids). The results of these experiments suggest that:

1. *The momentum model and professional forecasters have similar predictive power.*
2. *The momentum model has superior two-period ahead forecasts relative to the other forecasting methods.*
3. *The ARIMA models are relatively the worst forecast methods.*

The differential equation model is a closed system that involves the conservation of cash and shares. The interaction of these two forms of investment leads to a natural measurement of the liquidity of the market (*M/S*, the ratio of total cash to total shares in the market). We investigate the effect on prices by varying this liquidity measure and find: *The central tendency of prices in cash-rich experiments is significantly higher than prices in asset-rich experiments*.

The results of our experiments and modeling suggest that we should:

1. Allow traders to see the entire limit book in a low-liquidity environment to determine if more complete information will dampen expectations of price increases.
2. Allow for significant infusions of cash (accumulated dividends) or shares (accumulated stock dividends) into the market to determine if liquidity can rekindle momentum or stifle burgeoning bubbles.
3. Introduce another security into the model to see how momentum across markets affects the price dynamics.

The time series results of empirical data and laboratory data confirm the importance of momentum, which is often difficult to isolate in field data due to many factors that are difficult to quantify. The result that the predictive capacity of the momentum model is comparable to those of the best traders, and considerably better than the time series methods, is strong evidence that the basic concepts of momentum and liquidity have been incorporated into the model in a useful way.

The model can be further augmented by including information such as excess bids, which will likely result in better forecasts. For the portfolio manager, using such a model would be possible by postulating a fundamental value function of time and fitting the best parameters for a recent time period. The momentum model would then make some predictions for the near future that may be useful.

Notes

1. The initial distribution of shares and cash identified portfolio types as follows:

Trader Type	*Initial Cash*	*Initial Shares*	*Expected Value*
High Cash/Low Shares	$9.45	1	$13.05
Medium Cash and Shares	$5.85	2	$13.05
Low Cash/High Shares	$2.25	3	$13.05

2. We calculate amplitude as the difference between the highest deviation of mean contract price from its fundamental value and the lowest deviation of mean contract from its fundamental value. This value is then normalized by 360, the expected dividend value over fifteen periods.
3. Evidence with the capacity to short sell in this environment can be found in the section titled "Contracting Forms (Short Sales, Margin Buying, and Futures)" on page 30. Also, note that in Figure 4, the one-period expected dividend is only $0.16. This expected dividend is derived from a uniform distribution over the dividend values of ($0.00, $0.08, $0.16, $0.40).
4. Note that the nine public traders were endowed with twenty-one shares in total, while the "insiders" were endowed with seven owned shares plus a capacity to borrow up to twelve for short sales. Hence, the insiders' total selling capacity was nineteen shares, or 47.5% of the floating supply. Summing the net purchases by insiders (shown in brackets in figure 4)

across periods, we see that by period 11 insiders had sold all nineteen shares and thus had to become buyers to cover their short sells.

5. Although the advanced graduate students are "rational" in the sense of exhibiting common expectations that prices will reflect fundamental value, they fail to follow the dictates of "rational" behavior in two-person extensive bargaining (see McCabe and Smith [1999]). Consequently, we cannot say that training in rational theory predicts behavior consistent with the theory in all circumstances.
6. The *Wall Street Journal* (see Ip [1999]) reported that the 1998–1999 record-setting rise in blue-chip stock prices amid interest rate uncertainty, according to market analysts, can be traced to price momentum. In fact, more than 40% of fund managers use price and earnings momentum in their investment style.
7. Closed-end funds, unlike open-ended mutual funds, do not stand ready to redeem shares. The funds operate with a fixed number of shares outstanding and do not regularly issue new shares of stock. The stocks of these closed-end funds are actively traded in secondary markets.
8. In a commentary on a presentation of this example along with data from the laboratory, a discussant objected that the Spain Fund was a case of disequilibrium, and that it was also hard to obtain shares to sell short. But finding a sufficient supply of shares that can be borrowed for making short sales is a ubiquitous problem for any stock. Brokers limit the availability of shares for lending to short-sellers because those shares come from the floating stock held to cover shares purchased by their customers. These stocks are drained by sales to customers of other brokers, and hence limiting the potential demand will constrain the broker's short sales and reduce his risk. But that risk mounts as momentum traders bid up share prices. Hence the availability of borrowed shares is tightest when short-sellers most want them.
9. These experiments used a uniform price call market as the pricing mechanism (see Van Boening et al. [1993]. This mechanism was used because it has a single price prediction, as opposed to the double auction in which averages or closing prices would have to be selected.
10. SSW conducted some experiments in which subjects were asked to forecast the mean price for the next period with a monetary reward for the best forecaster across all periods. The consensus (mean) forecast results reveal that: 1) bullish capital gains expectations arise early in these experiments; 2) the mean forecast always fails to predict price jumps and turning points; and 3) the mean forecasts are highly adaptive, i.e., jumps in the mean price and turning points are only reflected in forecasts after a one-period lag. Also, individual forecasting accuracy was positively and significantly correlated with profits earned. These observations parallel the performance of professional forecasters, whose forecasts are reasonably accurate only when the forecasted measure does not change much, and who are notoriously inaccurate at forecasting turning points when you need them most (Zarnowitz [1986]).
11. For their experiments, SSW find R^2 values ranging from 0.04 to 0.63. In addition, the variances in the estimates of α are large, so that α is almost never significantly different than minus the one-period expected dividend.
12. The Wall Street Journal (Browning [1999]) reports that liquidity (defined as retirement savings) is a major factor in current stock price movements.

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Appendix
Dynamical System Derivation

A. Basic Model

We begin by reproducing the first few equations from the section titled "Mathematical Modeling of Momentum and Overreactions" on page 32. First, we have the dependence of supply and demand on the price trend:

$$\frac{d}{dt}(\log p) = F\left(\frac{d(p,p')}{s(p,p')}\right) \tag{5.3}$$

Next, we state the stock flow identities by letting B be the fraction of the total value of assets held in the form of shares ($1 - B$) is the fraction held in cash, so $B = pS/(pS + M)$, where we have a closed system containing M dollars and S shares:

$$d = k(1-B),\ s = (1-k)B, \text{ and } \frac{d}{s} = \frac{k}{1-k}\cdot\frac{1-B}{B} \tag{5.4}$$

All the behavioral features of this system will be defined in terms of k.

To develop the behavioral hypotheses concerning market decisions, let $p_a(t)$ denote the fundamental value of a share at time t. If k depended only on $p_a(t)$, we would have a generalization of the theory of price adjustment written in terms of the *finiteness* of assets and *delay* in taking action. However, if the rate k is specified through ζ defined as investor sentiment, the desire to accumulate shares, or preference for shares over cash, price can be adjusted based on investor perceptions.

To understand the dependence of ζ on the history of price change, consider the motivation of an investor who owns the security as it is undervalued but still declining. The choice available to this investor is to either sell or to wait in the expectation that those with cash will see the opportunity to profit by purchasing the undervalued security.

The total investor sentiment, or preference function, ζ, is expressed as the *sum* of the price trend ζ_1 and the price deviation from fundamental value ζ_2. In each case, the basic motivation is summed with a weighting factor that declines as elapsed time increases. This leads to the equations:

$$k(\zeta) = \frac{1}{2}\left[1+\tanh(\zeta_1+\zeta_2)\right] \tag{5.5}$$

$$\frac{d\zeta_1}{dt} = c_1\left[q_1 p^{-1}(t)\frac{dp(t)}{dt} - \zeta_1\right] \tag{5.6}$$

$$\frac{d\zeta_2}{dt} = c_2\left[q_2 \frac{p_a(t) - p(t)}{p_a(t)} - \zeta_2\right] \tag{5.7}$$

This system, together with (5.3) and (5.4), are a system of ordinary differential equations that can be studied numerically. Equation (5.5) has the property that k will range between (0, 1) as required by the definition of k, and will approach those bounds asymptotically—see Figure A-1.

Suppose the underlying behavioral sentiment impacts demand without delay, so that

$$\frac{d\zeta_1}{dt}, \frac{d\zeta_2}{dt} = 0$$

Then:

$$\zeta_1 = q_1 \frac{1}{p(t)} \frac{dp(t)}{dt} \tag{5.6'}$$

$$\zeta_2 = q_2\left(1 - \frac{p(t)}{p_a(t)}\right) \tag{5.7'}$$

$q_1, q_2, c_1,$ *and* c_2 are the only parameters in the system in addition to the scaling of time. Increasing q_1 increases the importance of trend-based investing and the amplitude of oscillations. Increasing q_2 increases the effect of investment based on price deviations from fundamental value. Numerical studies of these equations show that c_2 has very little effect on price evolution, and that large values of c_1 can lead to unstable oscillation.

B. Liquidity

Recall that we define liquidity as the ratio of cash to shares, $L=M/S$, which is measured in dollars per share. Assuming that the relative price changes linearly with excess demand, we have:

$$\frac{\tau_0}{p}\frac{dp}{dt} = \frac{d}{s} - 1 \tag{8.1}$$

for a time scale τ_0. Thus, from the identities in (5.4), we find

$$B = \frac{S \cdot p}{S \cdot p + M},\ 1 - B = \frac{M}{S \cdot p + M} \tag{8.2}$$

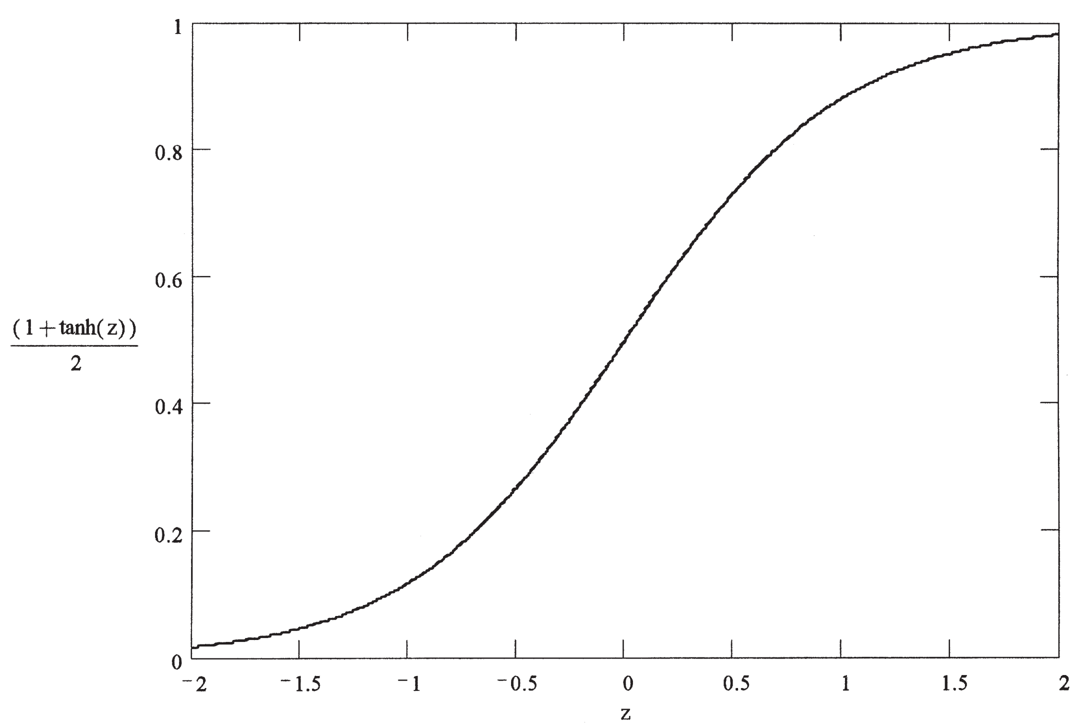
$$\frac{B}{1-B} = \frac{S \cdot p}{M} = \frac{p}{L} \tag{8.3}$$

so that $B^{-1}(1\text{-}B)p=L$ is time-invariant. Equation (8.3) leads to the derivative identity

$$\frac{dB}{dt} = B(1-B)\frac{1}{p}\frac{dp}{dt} \tag{8.4}$$

Note that there is no need for a time scale in this equation. Using the investor sentiment equations (5.6´) and (5.7´), and the assumption that the transition rate k is a weighted sum of the current derivative and the valua-

FIGURE A-1



tion discount, we can use (5.5) and a linear approximation of the hyperbolic tangent to get:

$$\frac{k}{1-k}=\frac{1+\tanh\left(\zeta_1+\zeta_2\right)}{1-\tanh\left(\zeta_1+\zeta_2\right)}\approx 1+2\frac{q_1\tau_0}{p(t)}\frac{dp}{dt}+ 2q_2\left(1-\frac{p(t)}{p_a(t)}\right) \tag{8.5}$$

Using (5.4) and (8.3) in the price equation, (8.1), results in

$$\frac{\tau_0}{p}\frac{dp}{dt}=\frac{k}{(1-k)}\frac{L}{p}-1 \tag{8.6}$$

and substituting for *k/(1-k)* using (8.5) leads to

$$\frac{\tau_0}{p(t)}\frac{dp}{dt}=\left[1+2q_1\frac{\tau_0}{p(t)}\frac{dp}{dt}+2q_2\left(1-\frac{p(t)}{p_a(t)}\right)\right]\cdot \frac{L}{p(t)}-1 \tag{8.7}$$

Note that the liquidity value, L, which represents the nominal value of all money in the system divided by the total number of shares, is a fundamental scale for price. In particular, p/L is the ratio of asset value to liquidity value. Multiply (8.7) by p/L and use $L_p := p/L$ and $L_{pa} := p_a/L$, so that we can rewrite Equation (8.7) as

$$\tau_0\frac{dL_p}{dt}=\left[1+2q_1\frac{\tau_0}{p}\frac{dp}{dt}+2q_2\left(1-\frac{p(t)}{p_a(t)}\right)\right]-L_p \tag{8.7$'$}$$

Since

$$\frac{dL_p}{dt}=\frac{dp}{dt}\frac{1}{L}$$

the terms in the bracket reduce to

$$1+2q_1\frac{\tau_0}{L_p}\frac{dL_p}{dt}+2q_2\left(1-\frac{L_p}{L_{pa}}\right)$$

Rearranging terms, we get:

$$\tau_0\left(1-\frac{2q_1}{L_p}\right)\frac{dL_p}{dt}=1+2q_2\left(1-\frac{L_p}{L_{pa}}\right)-L_p \tag{8.7$''$}$$

The left-hand side of (8.7´´) can be rewritten as

$$1+2q_2-\left(L+\frac{2q_2}{L_{pa}}L_p\right)$$

from which we can obtain:

$$\left(1-\frac{2q_1}{L_p}\right)\frac{dL}{dt}+\left(1+\frac{2q_2}{L_{pa}}\right)L_p=1+2q_2 \tag{8.8}$$

Assuming that

$$\frac{dL_p}{dt}\to 0 \text{ as } t\to 0$$

the first term in (8.8) vanishes so that the equation for equilibrium with $p=p_e$ is given by:

$$\left(1+\frac{2q_2}{L_{pa}}\right)L_{pe}=1+2q_2 \tag{8.8$'$}$$

If we multiply (8.8´) by L_{pa}, we obtain $(L_{pa} + 2q_2)L_{pe} = L_{pa} + 2q_2 L_{pa}$. Rearranging and grouping terms we get:

$$L_{pa}\left(1-L_{pe}\right)=2q_2\left(L_{pe}-L_{pa}\right) \tag{8.9}$$

Notice that if q_2 is large (that is, there is a strong sentiment to trade based on deviations from fundamental value), then $L_{pe} \sim L_{pa}$ so that fundamental value is attained. However, if q_2 is small, then $L_{pe}\sim 1$, so that $p\sim L$, which means that the liquidity value (total dollars divided by total number of shares) is attained as an equilibrium value. ***In the absence of clear information and attention to value, the price tends to gravitate to a natural value determined by the ratio of total cash to total quantity of asset.***

Solving (8.9) for q_2 yields (in the original units)

$$q_2=\frac{1-p_{eq}/L}{2\cdot\left(p_{eq}/p_a-1\right)} \tag{8.10}$$

For constant L_{pa}, Equation (8.8) has an exact solution that can be obtained by partial fractions and separation of variables. The solution is given by

$$L_p^{-2q_1/L_{pe}}\left|L_p-L_{pe}\right|^{1-2q_1/L_{pe}}=\mathrm{C}e^{s\tau} \tag{8.11}$$

where we have from (8.9)

$$L_{pe}:=\frac{1+2q_2}{1+2q_2/L_{pa}},\ s:=1+\frac{2q_2}{L_{pa}}$$

and C is a generic constant from the integration evaluated at some initial value, τ^*.

Using trading periods as discrete time intervals, the discrete version of (8.7) is:

$$\tau_0\left[1-\frac{2q_1}{p(t)}L(t)\right]\left(p(t+1)-p(t)\right)= \left[1+\frac{2q_2}{p_a(t)}L(t)\right]p(t)+\left(1+2q_2\right)L(t) \tag{8.12}$$

and similar discrete equations can be written as the analogs of the differential equations. A feature of the first-order equation is that it suppresses oscillations, unlike the higher-order equations.